Vol. 22, No. 10, October, 1961

Translation Published March, 1962

АВТОМАТИКА
и
ТЕЛЕМЕХАНИКА

SOVIET INSTRUMENTATION AND
CONTROL TRANSLATION SERIES

Automation and Remote Control

(The Soviet Journal *Avtomatika i Telemekhanika* in English Translation)

■ This translation of a Soviet journal on automatic control is published as a service to American science and industry. It is sponsored by the Instrument Society of America under a grant in aid from the National Science Foundation, continuing a program initiated by the Massachusetts Institute of Technology.

SOVIET INSTRUMENTATION AND CONTROL TRANSLATION SERIES

■ This translation of the Soviet Journal *Avtomatika i Telemekhanika* is published and distributed at nominal subscription rates under a grant in aid to the Instrument Society of America from the National Science Foundation. This translated journal, and others in the Series (see back cover), will enable American scientists and engineers to be informed of work in the fields of instrumentation, measurement techniques, and automatic control reported in the Soviet Union.

The original Russian articles are translated by competent technical personnel. The translations are on a cover-to-cover basis and the Instrument Society of America and its translators propose to translate faithfully all of the scientific material in *Avtomatika i Telemekhanika*, permitting readers to appraise for themselves the scope, status, and importance of the Soviet work. All views expressed in the translated material are intended to be those of the original authors and not those of the translators nor the Instrument Society of America.

Publication of *Avtomatika i Telemekhanika* in English translation started under the present auspices in April, 1958, with Russian Vol. 18, No. 1 of January, 1957. The program has been continued with the translation and printing of the 1958-1961 issues.

Transliteration of the names of Russian authors follows the system known as the British Standard. This system has recently achieved wide adoption in the United Kingdom, and is currently being adopted by a large number of scientific journals in the United States.

Readers are invited to submit to the Instrument Society of America comments on the quality of the translations and the content of the articles. Pertinent correspondence will be published in the Society's monthly publication, the ISA JOURNAL. Space will also be made available in the ISA JOURNAL for such replies as may be received from Russian authors to comments or questions by the readers.

1961 Volume 22 Subscription Prices:

Per year (12 issues), starting with Vol. 22, No. 1

General:	*United States and Canada*	$35.00
	Elsewhere	38.00
Libraries of nonprofit academic institutions:		
	United States and Canada	$17.50
	Elsewhere	20.50
Single issues to everyone, each		$ 6.00

1957 Volume 18, 1958 Volume 19, 1959 Volume 20, and 1960 Volume 21 issues also available. Prices upon request.
See back cover for combined subscription to entire Series.

Subscriptions and requests for information on back issues should be addressed to the:

Instrument Society of America
530 William Penn Place, Pittsburgh 19, Penna.

Translated and printed by Consultants Bureau Enterprises, Inc.

Automation and Remote Control

A translation of Avtomatika i Telemekhanika, a publication of the Academy of Sciences of the USSR

Vol. 22, No. 10 Russian Original Dated October, 1961 March, 1962

CONTENTS

SOVIET INSTRUMENTATION AND CONTROL TRANSLATION SERIES

Automation and Remote Control

A translation of Avtomatika i Telemekhanika, a publication of the Academy of Sciences of the USSR

Vol. 22, No. 10 — Russian Original Dated October, 1961 — March, 1962

CONTENTS

THE TREND TOWARD AUTOMATION

(IN ANTICIPATION OF THE 22ND CONGRESS OF THE CPSU)

(Editorial)

Translated from Avtomatika i Telemekhanika, Vol.22, No. 10,
pp. 1265-1268, October, 1961

In October of this year the 22nd Congress of the Communist Party of the Soviet Union is being convened. The congress will study the program for creating the foundations of communist society in our country.

N. S. Khrushchev vividly compared communist society with a cornucopia of the products of physical and spiritual labor which will be accessible to every member of society. Everyone shall have enough food, clothing, footwear, housing, books. N. S. Khrushchev says that the time is not far off when people in our country will work 3 to 4 hours a day on the basis of the further development of production, science and engineering, and on the basis of technical progress and the automatization of production. Men will control complex machines and mechanisms and in this short working day will produce appreciably more products than now; this will guarantee a fuller satisfaction of the requirements of the country and of all the requirements of each citizen of communist society.

Automation will help millions of Soviet workers make their work more perfect and intellectual, and will bring the level of development and the preparation of the worker closer to the level of the engineer. The creative forces and energy of Soviet man in the field of production will become ever more concentrated in scientific and research work; it will include the planning and creation of new effective means for achieving material welfare.

The reduction of working time due to automation is, as we know, the most important condition for true human freedom and will permit workers to devote greater attention to the development of their physical and mental capacities, to the enrichment of their spiritual surroundings, and to the all-round development of their talents.

In the USSR automation is becoming the task of the entire people. Because it is the most important part of technical progress the automation of production has become the general line of development in industry and is the center of attention of the Communist Party.

The scientific, engineering and organizational sides of automation form a unified whole. This makes it necessary to develop the proper forms of organization for scientific-research work and automation projects; it is also necessary to establish spheres of the effective application of automation and the preparation of cadres with the required qualifications.

The practice which has existed heretofore of automatizing individual production operations or parts of technological processes, notwithstanding the high engineering and economic effectiveness of individual projects, cannot assure the solution of the problem which has been posed. A qualitatively new result can be obtained only by introducing full automation which has not yet been widely developed. The basic obstacle to this development is the fact that many forms of existing equipment and technological processes are not suited to automation. In order to utilize all the possibilities and advantages of automation more fully and to make it one of the chief factors in the economic development of the country it is necessary to achieve a high level of production mechanization; the operating machines and the technology itself must be developed in accordance with the requirements and possibilities of highly efficient automation. New technological processes and aggregates must be designed in such a way that the automatic control systems are not mechanical auxiliaries but organic parts of the process and aggregates. Automation must encompass all sections of production with a unified automatic control system that assures integrated operation of all aggregates in optimal modes, the complete mechanization of the technological processes, and a highly organized flow of production.

For this purpose it is necessary to employ new forms for coordinating the work of scientists and research and design organizations. One such form is the complex development of plans for proposed projects involving new automatic installations in the basic sections of industry. Such plans must take into account the interrelation of the problems involving automatic control, the newest technology and equipment, a high degree of mechanization, and adequate power resources; it must involve the joint efforts of engineers, machine builders and automation specialists.

The plans developed by leading specialists in various scientific and engineering fields must provide a clear concept of the automated factories of the future and must define those scientific engineering problems which must be solved so that the automated factories can be constructed in a brief time.

At the present time the science of automatic control has achieved considerable successes. The general principles and laws which have been discovered for designing automatic control systems, as well as the engineering means for realizing these systems, have made it possible to introduce automatic control into industry and to assure not only an appreciable improvement in the engineering-economic production indices but to create a number of new processes which were inconceivable under conditions of manual control (atomic power installations, continuous rolling mills with a rolling speed exceeding the speed of an express train, installations for the annealing and gasification of materials in a boiling layer, etc.).

However, the functions which a human being performs in controlling technological processes are immeasurably more expensive than those which modern automatic control systems perform. The automatic control systems of the future must eliminate the effect of human subjective factors on the engineering-economic indices of the process; they must process an enormous amount of information, analyze the flow of the process, and plan and program its control in advance.

The design of such automatic systems must be based on the wide utilization of computer engineering techniques which will make it possible to remember a greater amount of data, to solve complex logic problems automatically, and to perform various types of calculations at a great speed. For comparatively simple deterministic technological processes we can build control systems which are program control systems.

For the more complex processes for which it is impossible to formulate a control program in advance, a program will be sought automatically during the course of the process itself on the basis of retrieved information and approximate data on the properties and characteristics of the process which is stored in the system. In such automatic systems self-adaptive and learning methods will find the widest application. Therefore it is necessary to expand the work on the theory and methods for choosing the optimal automatic control strategy under conditions of incomplete a priori information on the characteristics of the objects and under conditions where noise is present. For this purpose it is necessary to develop and apply new mathematical techniques (algorithm theory, the theory of operations, games theory, information theory, etc.).

It is necessary to expand work on the development of the theory and principles of designing self-adaptive and other automation and remote-control systems which are capable of adapting to changing external conditions.

In designing complex automatic control systems the study of the characteristics and properties of the automatized processes acquires exceptional importance. At the present time such a study is being performed using extremely primitive means which require a great deal of time expended by the researchers and do not yield reliable information. As a result it is sometimes necessary to perform a prolonged (sometimes lasting many years) refinement of the designed automatic control systems. It is necessary to develop the methods and the automatic equipment for studying the dynamic characteristics and energy properties of controlled objects, and to perform a statistical analysis of the processes which occur in them and the perturbations which act on them. It is also necessary to develop methods for determining the optimal parameters of new automatic machines, production lines, and complexes on the basis of fundamental investigations of the kinematics, dynamics and accuracy of the individual mechanisms and machines; further, we must develop the theoretical foundations of automatic technological and transportation processes.

In the near future automatic devices which are used not only for automatizing production processes but also for automatizing elements of mental work in operations which involve appraisal, planning, design, and scientific research will begin to find wide application.

In order to solve this problem it is necessary to develop the study of the mental activity of a human being in conjunction with physiologists; in particular, it is necessary to investigate the processes involved in the solution of problems by the human brain (in such investigations, of course, the examination of all possible variants is impossible). It is necessary to develop the principles governing the design and theory of programming control machines and to take up the investigation of the physiological features of the human being for the purpose of developing more efficient control systems.

In order to produce effective control systems we must find new principles governing the design of automation elements by basing ourselves on the newest achievements in physics and chemistry, including the use of physico-

chemical processes and reactions, intra-molecular and intra-atomic processes, nuclear and other types of radiation, etc.

It is necessary to develop new and highly reliable automatic control devices which are based on the use of semiconductor, magnetic and other contactless elements. Pneumatic automatic control systems, which do not contain moving parts and use the interaction of a jet, must be applied as widely as possible. We must develop microelements for automatic devices, including logic and memory elements which have ultra-small dimensions and assure high speed of response, high reliability, and are easy to manufacture; on the basis of these elements we can develop microdevices which simulate nerve tissues.

The combination of logic and computing operations in complex automatic control systems is being successfully solved through the application of digital techniques. Therefore it is necessary to assure the widest application of automation devices which employ digital engineering and feature the aggregate principle of design. This, in turn, requires the replacement of analog transducers by transducers which represent the values of the measured quantities directly in digital form.

It is necessary to perform investigations which assure transition to a common state-administered system of engineering means for automation which is based on the wide utilization of digital engineering elements.

We must develop the theoretical foundations for designing automatic control systems that assure processing of the retrieved information in a form that is convenient for the use of the servicing personnel.

Machines are required for the automatization of the processes involved in synthesizing and analyzing the structures of automatic devices, for planning and designing, for realizing complex programs for checking, testing, adjusting, and assembling. It is necessary to develop units which perform the processes involved in analyzing and synthesizing the structures of complex control devices by machine.

Remote control units which make it possible to incorporate machines and technological aggregates distributed over wide areas into a single controlled complex will be developed extensively (units for use at oil-well installations, irrigation systems, power systems, gas- and water-supply systems, transportation systems, etc.).

An important condition governing the development of automatization is the assurance of operational reliability of automatic and remote control systems. This requires extensive expansion of work in the field of reliability.

Here we need to develop both the theory of structural reliability and the theory of the reliability of devices and their elements as a function of their characteristics, operating conditions and modes of utilization. It is necessary to develop engineering methods for computing and improving the reliability of automation and remote control systems, including methods for constructing highly reliable devices using elements with low reliability.

The achievement of the enumerated measures will be a great contribution to creation of the material and engineering base of communist society.

Soviet specialists on automation are inspired by the new draft program of CPSU which will be discussed at the forthcoming 22nd Congress of the CPSU; they will apply all their creative forces, knowledge and energy to achieve the problem posed before them.

VICTOR SERGEEVICH KULEBAKIN

(ON HIS SEVENTIETH BIRTHDAY)

(Editorial)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1269-1272, October, 1961

On October 30, 1961 the outstanding Soviet scientist in the field of electrical engineering and automation, Stalin Prize laureate, Academician Victor Sergeevich Kulebakin will reach his seventieth birthday and the anniversary of 45 years of creative scientific and pedagogical activity.

Victor Sergeevich was born October 30, 1891 in the family of a rural teacher. In 1909 he entered the Moscow Higher Technical School where he specialized in the field of internal combustion engines and electrical engineering. In his advanced courses he heard a series of lectures by N. E. Zhukovskii on aviation and made a detailed study of the theory of heat engines; he also performed a series of research assignments in the field of mercury rectifiers and electrical machines.

V. S. Kulebakin was an initiator and active participant in the foundation of new scientific establishments (the State Experimental Electrical Institute, later renamed the V. I. Lenin All-Union Power Institute; the Institute of Automation and Remote Control of the Academy of Sciences, USSR) and of new fields of industry and forms of production (instrument design, the production of special instruments, etc.). He participated actively in the development of the Lenin electrification plan for the country ("GOÉLRO") and performed tasks assigned by the "GOÉLRO" commission.

A great contribution to the realization of the "GOÉLRO" plan was made by a large group of scientists among whom V. S. Kulebakin stood in the front ranks.

It is precisely to participation in the solution of this great task of the economy of the country that one of the basic fields of activity of V. S. Kulebakin was devoted.

Working in the electrical engineering laboratory of the Moscow Higher Technical School and later in the laboratories of the All-Union Electrical Engineering Institute and the Moscow Power Institute, he conducted a number of research projects in the field of electrical machinery, electrical measurements, electrical equipment, electrical power, etc.

In these fields V. S. Kulebakin has written scientific papers devoted to computations and further theoretical developments in the fields of stationary and transient processes in dc electrical machinery, synchronous and asynchronous electrical machines and transformers. He proposed a number of new designs for electrical machines and investigated new systems. For his development of an electrical mining locomotive with a capacitor induction motor V. S. Kulebakin along with other participants in this project was awarded a Stalin Prize.

Of great scientific value are the papers written by V. S. Kulebakin on the theory of cross-field rotary amplifiers, on the pulse method of controlling the speed of electric motors, and other problems. Important papers were written by V. S. Kulebakin on the kinetics involved in the excitation of electrical machines; these papers form the basis for a new branch of the theory of electrical machines which has become especially significant in recent years due to the wide use of rotary automation and the development of new large power systems.

Much space in V. S. Kulebakin's papers is occupied by automatic control of electrical machines. He developed the theory and methods of computation for various types of voltage regulators. In his monograph "Electrical Equipment," which has sold out in several editions, a systematic presentation of the methods for computing and designing starting and control equipment is given. His fundamental work on the computation and design of starting and controlling rheostats was awarded a prize by the People's Commisariat of Education.

In a number of his papers V. S. Kulebakin developed the theory and methods of computing devices for the electrical ignition of internal combustion engines; these papers provided the first scientific bases for this field of equipment construction. In this field he is one of the greatest specialists, and his papers are still widely used in practice.

V. S. Kulebakin devoted a great deal of work, energy and initiative to solving problems in aviation electrical equipment; he studied this field from the very first days of his engineering and scientific career. His research encompasses basic problems in this complex and varied branch of engineering; in particular, he dealt with problems of lighting (illumination for night flights, reflection of light from terrestrial coverings, the reflection of light from rotating propellers, the computation of the illumination of open spaces).

A very important and essential part of the work performed by Victor Sergeevich Kulebakin is in the field of automatic control theory. His most significant work dealt with the perturbed movement of automatic control systems in the presence of external signals f(t). In the early period of development of automatic control theory V. S. Kulebakin pointed out the importance of the analysis and synthesis of automatic control systems while taking into account perturbations f(t) which act continuously. In 1940 V. S. Kulebakin cited the general conditions for determining the basic parameters of automatic controllers in the presence of external perturbations. In treating the characteristics of transient responses x(t) in automatic control systems V. S. Kulebakin first proposed the use of integral predictions for processes. In 1940 he established the relationship between the integral $\int_0^\infty x\,dt$ and the parameters of a control system (in particular, their static accuracy). As we know, later on the use of the integral prediction method in automatic control theory proved to be exceptionally fruitful.

In general control theory the integral predictions proposed by V. S. Kulebakin were developed for the case where the mean-square estimate of system accuracy is to be determined, so that these predictions can be used as criteria in the general theory of automatic control system optimalization. Victor Sergeevich Kulebakin was one of the few scientists and researchers who came forward 20 years ago with an energetic defense of frequency methods in automatic control theory.

A natural development of the indicated work by Kulebakin was his work in the field of the general theory of automatic control system invariance. At the present time the general front of research in this field, both in the Soviet Union and abroad, is extremely extensive and V. S. Kulebakin is undoubtedly the leader of this scientific school in automatic control theory. V. S. Kulebakin was able to draw in the most outstanding scientists and specialists to work on his problem; these included Academician N. N. Luzin who established the mathematical conditions and the theorems of absolute invariance and invariance with an accuracy of up to ε in the theory of differential equations. Later on V. S. Kulebakin used N. N. Luzin's theorem as the basis for proving that in a multiple-loop system that is subjected to the simultaneous effect of several perturbations it is possible to create conditions such that any coordinate x_i becomes independent of one or several perturbations f(t) (i.e., the achievement of so-called selective invariance). In the field of invariance theory V. S. Kulebakin has achieved results such as the solution of the problem involving the applicability of the principle of absolute invariance in real physical systems, the general nature of problems of autonomous control and selective invariance, further development of the theory of combined automatic control systems, the forms and conditions governing invariance in various automatic control systems and its utilization in programming and optimalizing mathematical and control machines. In testing the problem involving the behavior of continuously perturbed automatized linear systems and depicting the functions by means of the integrals of differential equations, V. S. Kulebakin produced the theory of the K(D) transform for functions satisfying the condition $K(D)f(t) = 0$ for which $K(D) \neq 0$; $f(t) \neq 0$. The indicated condition represents a special form of invariance for automatic control systems that has a great practical significance.

In summarizing this work V. S. Kulebakin demonstrated that the principle of total or partial invariance is applicable to real physical devices. The established invariance criteria make it possible to devise methods for the accurate computation of physical and structural parameters for these devices, and automatic systems designed according to the principles of invariance are refined and rational. In developing this point of view, V. S. Kulebakin frequently put forward the idea of designing automatic systems and production objects in direct conjunction, and of assigning the properties of controllability to the production object proper while altering these properties in the direction of self-adaptation and automatizability.

V. S. Kulebakin is not only an outstanding scientist but also a talented teacher who is able to organize a faculty tightly around him and to inculcate interest in and love for science in young people.

In his long years of professorial activity at the Moscow Power Institute and the Zhukovskii All-Union Higher Aviation Institute Victor Sergeevich developed a number of new courses: a general course in electrical equipment,

starting and control equipment, automatic control of electrical machinery, aviation electrical engineering, the electrical equipment of airplanes, etc. V. S. Kulebakin also compiled textbooks and various other books on most of the subjects mentioned above. As an able and experienced teacher V. S. Kulebakin devotes a great deal of energy to the education of young scientific workers and students and supervises their work. Many of them have already become outstanding figures in the national economy, scientists and engineers, candidates and doctors of the technical sciences, professors, and Lenin Prize Laureates.

Victor Sergeevich is doing a great deal of government and organizational work: for eight years he was a member of the Higher Degree Commission of the All-Union Committee of Higher Education of the USSR; he also headed the Methods Commission of the Power Engineering Universities (G.U.V.U.Z. V.S.N.Kh.), worked as a member of the Commission for the Mobilization of the Ural Resources during the Great Patriotic War, participated in the work of the Commission on Developing the GOÉLRO Plan before the war and on the Commission on 100-Cycle Current attached to the Council of Labor and Defense, and on the Commission of the Division of Technical Sciences of the Academy of Sciences USSR, on the electrification of railway transport. From 1940 on V. S. Kulebakin was a member of the Council of Engineering and Economics Experts of the State Plan of the USSR for many years where he participated in scientific-engineering planning for the development of the national economy.

From 1951 on V. S. Kulebakin took an active part in the Academy of Sciences, USSR, in providing scientific help and cooperation to the builders of large hydroelectric power stations; he headed the brigade of scientists of the Academy of Sciences, USSR, on the Stalingrad hydroelectric complex.

V. S. Kulebakin is an active public figure. From 1953 through 1959 he was a deputy of the Moscow City Council of Workers Deputies. V. S. Kulebakin is one of the leading organizers and active participants of a number of scientific conferences and conventions. He is a member of the editorial board of leading scientific journals of the USSR and is the editor-in-chief of the journal, "Izvestiya Akad. Nauk SSR, Otdel. Tekh. Nauk."

The activity of V. S. Kulebakin as a member of the Bureau of the Division of Technical Sciences of the Academy of Sciences USSR and as a member of the presidium of the Editorial and Publishing Council of the Academy of Sciences USSR, etc. is well known.

V. S. Kulebakin is the author of more than 300 scientific and engineering papers published in the form of books, monographs, textbooks, and articles. Many works have been published under the editorship of V. S. Kulebakin.

The election of V. S. Kulebakin as a corresponding member of the Academy of Sciences USSR in 1933, and as an Academician in 1939 was an acknowledgment of his great services to Soviet science.

In 1942 Victor Sergeevich was awarded the title of Major General of Aviation Engineering. The activity of V. S. Kulebakin has been marked by high government awards: two orders of Lenin, the Order of the Red Banner, the Order of the Red Banner of Labor, three Orders of the Red Star, two Orders of the Badge of Merit, and six medals. In 1961 V. S. Kulebakin was awarded the honorary title of Distinguished Scientist and Engineer of the RSFSR.

In honoring Academician V. S. Kulebakin on his 70th birthday the editorial board of "Avtomatika i Telemekhanika" wishes him long life and further success in his scientific work for the good of our country.

ANALYTIC DESIGN OF CONTROLLERS IN SYSTEMS WITH RANDOM ATTRIBUTES

II. EQUATIONS OF OPTIMUM SOLUTIONS. APPROXIMATE SOLUTIONS

N. N. Krasovskii and É. A. Lidskii

(Sverdlovsk)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1273-1278, October, 1961

Original article submitted March 18, 1961

The general approach, as described in [1], is used to derive the equations which determine the optimum control rule. The method of obtaining approximate solutions is described.

1. Equations of the Optimum Lyapunov Function v^0 and of the Optimum Control Rule ξ^0

1. The principles to be used in determining the optimum control rule were formulated in [1], the latter being such that it minimizes the integral criterion of quality:*

$$I_\xi = \int_0^\infty M\{\omega[x(t), \xi(t)] / x_0, \eta_0, t_0 = 0\}\, dt = \min_\xi \tag{1.1}$$

of the stochastic system under control

$$\frac{dx_i}{dt} = \varphi_i[x_1, \ldots, x_n, \eta(t), \xi], \tag{1.2}$$

$$\xi = \xi[x_1, \ldots, x_n, \eta]. \tag{1.3}$$

In accordance with these equations, ξ^0 is determined by the condition

$$\left[\frac{dM\{v^0\}}{dt} + \omega\right]_{\xi^0} = \min_\xi \left[\frac{dM\{v^0\}}{dt} + \omega\right]_\xi = 0, \tag{1.4}$$

where v^0 is the positive-definite optimum Lyapunov function (see (4.1) and (4.2) of [1]). The proof is given in Appendix I. We shall now obtain partial differential equations for v^0 and ξ^0 which follow from (1.4).

2. The derivative $dM\{v\}/dt$ at the point (x, η) can in view of the equations (1.2) and (1.3), be written as **

$$\frac{dM\{v\}}{dt} = \sum_{i=1}^n \frac{\partial v(x,\eta)}{\partial x_i} \varphi(x, \xi, \eta) + \int_{-\infty}^{\infty} v(x, \beta)\, d_\beta\, q(\eta, \beta) -$$

$$- q(\eta)\, v(x, \eta) + \frac{\lambda}{2} \sum_{i,j=1}^n \frac{\partial^2 v(x, \eta)}{\partial x_i \partial x_j} k_{ij}\mu_i\mu_j\sigma_i\sigma_j. \tag{1.5}$$

---

* The concepts and notation used are as introduced in [1].

** In (1.5) the integral is understood to be a Stieltjes integral [2].

In the particular case of $\eta(t)$ assuming only a finite number of values $\eta(t) = \{\alpha_1, \ldots, \alpha_k\}$, we have (at the point x, α_l)

$$\frac{dM(v)}{dt} = \sum_{i=1}^{n} \frac{\partial v(x, \alpha_l)}{\partial x_i} \varphi_i(x, \xi, \alpha_l) + \sum_{j \neq l}^{k} p_{lj}(v(x, \alpha_j) - v(x, \alpha_l)) + \frac{\lambda}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v(x, \alpha_l)}{\partial x_i \partial x_j} k_{ij} \mu_i \mu_j \sigma_i \sigma_j. \tag{1.6}$$

In the case when the function $q(\alpha, \beta)$ has a density $p(\alpha, \beta)$, this can be written as

$$\frac{dM(v)}{dt} = \sum_{i=1}^{n} \frac{\partial v(x, \eta)}{\partial x_i} \varphi_i(x, \xi, \eta) + \int_{-\infty}^{\infty} v(x, \beta)\, p(\eta, \beta)\, d\beta - q(\eta)\, v(x, \eta) + \frac{\lambda}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v(x, \eta)}{\partial x_i \partial x_j} k_{ij} \mu_i \mu_j \sigma_i \sigma_j. \tag{1.7}$$

The derivation of the formulas (1.5)-(1.7) is given in Appendix II.

3. The equation

$$\sum_{i=1}^{n} \frac{\partial v(x, \eta)}{\partial x_i} \varphi_i(x, \xi, \eta) + \int_{-\infty}^{\infty} v(x, \beta)\, d_\beta q(\eta, \beta) - q(\eta)\, v(x, \eta) + \frac{\lambda}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v(x, \eta)}{\partial x_i \partial x_j} k_{ij} \mu_i \mu_j \sigma_i \sigma_j + \omega[x, \xi, \eta] = 0 \tag{1.8}$$

(as well as its special forms which can be obtained from (1.6) or (1.7)) provides the first equation for the determination of $v^0(x, \eta)$ and $\xi^0(x, \eta)$. The other equation for v^0 and ξ^0 is derived by differentiating (1.8) with respect to ξ because, according to the condition (1.4), the left-hand side of (1.8) attains its minimum at $\xi = \xi^0$: *

$$\sum_{i=1}^{n} \frac{\partial v(x, \eta)}{\partial x_i} \frac{\partial \varphi_i(x, \xi, \eta)}{\partial \xi} + \frac{\lambda}{2} \sum_{i,j=1}^{n} \frac{\partial^2 v(x, \eta)}{\partial x_i \partial x_j} k_{ij} \sigma_i \sigma_j \frac{\partial(\mu_i \mu_j)}{\partial \xi} + \frac{\partial \omega}{\partial \xi} = 0. \tag{1.9}$$

The optimal functions v^0 and ξ^0 satisfy, therefore, the equations (1.8) and (1.9), and moreover this solution is adopted where v^0 is a positive-definite function.

2. Arriving at Approximate Solution of the Problem

The equations (1.8) and (1.9) prove too difficult to solve in the general case. This is also the case when attempting to find a solution which would ensure the bringing of the trajectory to the point x = 0 (i.e., such that its v^0 will be a positive-definite function). The following method, however, which leads to approximate solutions, is presented. Instead of the equations (1.2) and (1.3) we shall consider the auxiliary system

$$\frac{dx_i}{dt} = \psi_i[x, \eta, \vartheta], \quad \xi = \xi(x, \eta, \vartheta), \tag{2.1}$$

where the introduced parameter ϑ is such that when $\vartheta = 0$ the minimum of the functional

$$\int_0^\infty \varepsilon(x, \xi, \vartheta)\, dt$$

* Should ξ^0 be submitted to an additional constraint (for instance $|\xi^0| \leq 1$), the minimum in (1.4) should be reached with this condition also taken into account.

is attained in a straightforward manner and also when ϑ varies from zero to unity, the functions ψ and ε approach in a continuous manner the functions φ_1 and ω respectively. The optimal Lyapunov function $v^0(x, \eta, \vartheta)$ and the optimal control rule $\xi^0(x, \eta, \vartheta)$ should satisfy the equations (1.8) and (1.9) for every ϑ; therefore, differentiating the latter with respect to ϑ one is able to obtain the equations which describe the variation of the solutions ϑ^0 and ξ^0 with ϑ. Availing oneself of this device may in a number of cases facilitate the solution. In particular, one can try to find v^0 and ξ^0 in the series form

$$v^0 = \sum_k a_k(\vartheta)\,\zeta_k(x, \eta), \quad \xi^0 = \sum_k b_k(\vartheta)\,\zeta_k(x, \eta) \tag{2.2}$$

in terms of some functions $\zeta_k(x, \eta)$ in such a way that the equations (1.8) and (1.9) are satisfied not exactly by the finite aggregate (2.2) but approximately (for example, in the best mean-square approximation sense). The conditions of such approximation lead to equations describing the variation of the coefficients $a_k(\vartheta)$ and $b_k(\vartheta)$ with respect to ϑ. The method is also notable for the fact that if one starts with the stable solution of a problem for $\vartheta = 0$ and then the problem and also its solution are continuously varied, one is able to obtain the same branch of the solution for the equations (1.8) and (1.9) which, for every ϑ, will give solutions ensuring that the trajectories of the transient process attain the prescribed motion $z_0(t)$ (when $x(t) = 0$).

3. Example

A simple example will illustrate the described methods for obtaining a solution. Let us examine the selection of a regulator $\xi^0 = \xi^0(x, \eta)$ for a second-order system whose damping $\eta(t)$ is known only probabilistically:

$$\frac{d^2x}{dt^2} + \eta(t)\frac{dx}{dt} + x = m\xi + \gamma(x) \quad (m = \text{const}), \tag{3.1}$$

or in an equivalent form of simultaneous equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 - \eta(t)\,x_2 + m\xi + \gamma_2(x). \tag{3.2}$$

We shall confine ourselves to the simple case of $\eta(t)$ assuming only two values: $\eta = \eta_1$ and $\eta = \eta_2$ with the transition probabilities $P[\eta_i \to \eta_j$ within time $\Delta t] = p_{ij}\Delta t + o(\Delta t)$. The random disturbance γ_2 shall be characterized by the frequency $\lambda > 0$, by the function $\mu_2(x, \xi, \eta) = \mu x_2$ (μ = const) and by the random quantity ν_2 whose variance is σ_2^2 ([1], §2). It is required under these conditions to find the control rule

$$\xi^0 = \xi^0(x_1, x_2, \eta) \tag{3.3}$$

such that whatever the initial conditions x_{10}, x_{20}, $\eta_0 = (\eta_1$ or $\eta_2)$, and $t_0 = 0$, a probabilistically stable transient process should occur in the systems (3.2) and (3.3) which will minimize the mean-square error

$$I(x_{10}, x_{20}, \eta_0) = \int_0^\infty M\{x_1^2(t) + x_2^2(t) + \xi^2(x_1(t), x_2(t), \eta(t))\}\,dt = \min_\xi. \tag{3.4}$$

We shall endeavor to find the optimal Lyapunov function when the latter is a quadratic form

$$v_0 = b_{11}(\eta)\,x_1^0 + 2b_{12}(\eta)\,x_1x_2 + b_{22}(\eta)\,x_2^2.$$

In such a case the equations (1.8) and (1.9) take the form

$$\begin{gathered}2\left(b_{11}^{(l)}x_1 + b_{12}^{(l)}x_2\right)x_2 + 2\left(b_{12}^{(l)}x_1 + b_{22}^{(l)}x_2\right)\left(-x_1 - \eta_l x_2 + m\xi\right) + \\ + p_{ls}\sum_{i,j=1}^{2}\left(b_{ij}^{(s)} - b_{ij}^{(l)}\right)x_ix_j + \lambda b_{22}^{(l)}\mu^2 x_2^2\sigma_2^2 + \sum_{i=1}^{2} x_i^2 + \xi^2 = 0, \\ (l = 1,2;\ \ s = 1,2;\ \ l \neq s), \quad \left(b_{ij}^{(l)} = b_{ij}(\eta_l)\right),\end{gathered} \tag{3.5}$$

$$\left(b_{12}^{(l)}x_1 + b_{22}^{(l)}x_2\right)m + \xi = 0. \tag{3.6}$$

By eliminating ξ from the equations, and afterwards by comparing the coefficients of the products $x_i x_j$ we obtain a system of quadratic algebraic equations for the coefficients $b_{ij}^{(l)}$ $(l = 1, 2)$.

APPENDIX I

An outline is given for the proof showing that the optimal Lyapunov function v^0 (1.4) represents the solution. The asymptotic stability in probability [3] of the solution $x = 0$ under the condition that v^0 is a positive-definite function results from the theorems in [3]; these are proved in [3] when the function $\eta(t)$ is a random function of a less general type; nonetheless this discussion remains valid for the variable $\eta(t)$.

We shall show that it follows from the condition (1.4) that (1.1) is minimized. It is assumed that the functions v and ξ^0 are sufficiently smooth to ensure the existence of solutions to the equations (1.2) and (1.3) and also to enable one to use the derivative $dM\{v\}/dt$ and to perform the transformations applied subsequently. By averaging over the random quantities $x_i(x_0, \eta_0, t_0, \eta, \xi^0)$, $\eta(\eta_0, t_0, t)$, we obtain from (1.4) the formula *

$$\left(\frac{dM\{v(x(t), \eta(t))/x_0, \eta_0, t_0 = 0\}}{dt}\right)_{\xi^0} = M\{(dM\{v(x(t), \eta(t))\}/dt)_{\xi^0}/x_0, \eta_0, t_0 = 0\} =$$

$$= -M\{\omega[x(t), \xi^0(t)]/x_0, \eta_0, t_0\}, \tag{I}$$

which implies that $M\{v(x(t), \eta(t))/x_0, \eta_0, t_0 = 0\}_{\xi_0}$ decreases with the increase of t. Integrating with respect to t from $t = 0$ to $t = T$ we obtain

$$M\{v(x(T), \eta(T))/x_0, \eta_0, t_0 = 0\}_{\xi^0} - v(x_0, \eta_0) =$$

$$= -\int_0^T M\{\omega[x(t), \xi^0(t)]/x_0, \eta_0, t_0 = 0\}\,dt. \tag{II}$$

From (II) we draw the conclusion that the integral is convergent when $T \to \infty$. But, in view of ω being non-negative this means that $\lim M\{\omega\}_{\xi^0} = 0$ when $t \to \infty$, that is, that in accordance with the condition b) [1] the monotonic function $M\{v(x(t), \eta(t)/x_0, \eta_0, t_0 = 0\}_{\xi^0}$ approaches zero when $t \to \infty$. Therefore,

$$v(x_0, \eta_0) = \int_0^\infty M\{\omega[x(t), \xi^0(t)]/x_0, \eta_0, t_0 = 0\}\,dt, \tag{III}$$

that is, the expression $I_{\xi^0}[x_0, \eta_0]$ is finite, and $I_{\xi^0} = v$.

We shall now use the indirect proof method and assume that there exists a function $\xi^*[x, \eta] \neq \xi^0[x, \eta]$ such that when $\xi = \xi^*$ in (1.3) and (1.4), the solutions $\{x(t), \eta(t)\}_{\xi^*}$ produce the inequality

$$I_{\xi^*}[x_0, \eta_0] < I_{\xi^0}[x_0, \eta_0] \tag{IV}$$

for some initial conditions x_0, η_0 (at $t_0 = 0$).

The condition (1.4) implies that

$$\left(\frac{dM\{v\}}{dt}\right)_{\xi^*} \geqslant -\omega[x, \xi^*].$$

By averaging the inequality over the random quantities $x(x_0, \eta_0, t_0, \eta, \xi^*)$, $\eta(\eta_0, t_0, t)$ and integrating subsequently with respect to t we obtain the inequality

$$M\{v(x(T), \eta(T))/x_0, \eta_0, t_0 = 0\}_{\xi^*} - v(x_0, \eta_0) \geqslant$$

$$\geqslant -\int_0^T M\{\omega[x(t), \xi^*(t)]/x_0, \eta_0, t_0 = 0\}\,dt.$$

* The index ξ^0 signifies that the quantities are evaluated when the control rule is $\xi = \xi^0[x, \eta]$.

As the integral on the right-hand side of the inequality is convergent when $T \to \infty$ (in view of the assumption IV), by an analogous argument we conclude that the following inequality is valid:

$$v(x_0, \eta_0) \leqslant \int_0^\infty M\{w[x(t), \xi^*(t)] / x_0, \eta_0, t_0 = 0\}\, dt = I_{\xi^*}[x_0, \eta_0]. \tag{V}$$

But the inequality (V) is not consistent with the formula (III) and the assumption (IV). Thus the optimality of ξ^0 is proved.

APPENDIX II

The fundamental idea with regard to the derivation of the formula for $dM\{v\}/dt$ is outlined. The case is considered when $\eta(t)$ assumes a finite number of values only. The derivative $dM\{v\}/dt$ can be obtained by proceeding to the limit, as described by $dM\{v\}/dt = \lim M\{\Delta v\}/\Delta t$ when $\Delta t \to 0$. When deriving the formula, the terms of the order of smallness higher than Δt can be ignored. Therefore, if the quantities $x(t) = x$, $\eta(t) = \alpha_i$ occur at the instant t, it is sufficient to take into account during the time interval $t \leq \tau < t + \Delta t$ the following events: A – that the quantity $\eta(t)$ preserves its value α_i, $P(A) \approx 1 - \left(\sum_{j \neq i}^{k} p_{ij}\right)\Delta t$; B_j – that the quantity $\eta(t)$ changes its value $\eta(t) = \alpha_i$ precisely once, and at the instant $t + \Delta t$ we have $\eta(t + \Delta t) = \alpha_j$, $P(B_j) = \Delta t\,(p_{ij})$; $C(\nu_1, \ldots, \nu_n)$ — that within the time interval Δt an impulse $\mu_i \nu_i \delta$ $(i = 1, \ldots, n)$ occurs at the input, that is, the coordinates x_i change step-wise by $\Delta_u x_i \approx \mu_i(x, \eta, \xi)\nu_i$, $P(C) \approx \lambda \Delta t$. The events A and $\sum_{j \neq i}^{k} B_j$ are complementary, and the events $C(\nu_1, \ldots, \nu_n)$ are independent of A or of B_j. We denote by RQ the common part of the events R and Q; by Q^{-1} the event complementary to Q; by Δv_Q the increment of the function v along the trajectory of the system (1.2), (1.3) provided the event Q has taken place. We have

$$\begin{aligned} M\{\Delta v\} &\approx \underset{\nu}{M}\{\Delta v_{AC}\} P(AC) + \sum_{j \neq i}^{k} \Delta v_{B_j} P(B_j) + \Delta v_{AC^{-1}} P(AC^{-1}) \approx \\ &\approx \underset{\nu}{M}\left[\sum_{s=1}^{n} \frac{\partial v(x, \alpha_i)}{\partial x_s} \Delta_u x_s + \frac{1}{2} \sum_{s,\,l=1}^{n} \frac{\partial^2 v(x, \alpha_i)}{\partial x_l\, \partial x_s} \Delta_u x_s \Delta_u x_l\right] \lambda \Delta t + \\ &+ \sum_{j \neq i}^{k} [v(x, \alpha_j) - v(x, \alpha_i)]\, p_{ij} \Delta t + \sum_{s=1}^{n} \frac{\partial v(x, \alpha_i)}{\partial x_s} \varphi_s(x, \alpha_i)\, \Delta t \end{aligned} \tag{VI}$$

with an accuracy to the order of magnitude $o(\Delta t)$.

The formula (VI) is obtained in the following way. The second term is easily obtained and does not require any comments; the third term follows from the well-known rules which determine the increment of the Lyapunov function for ordinary differential equations [4] in view of the fact that for $\eta = \alpha_i$ = const the equations (1.2) become ordinary differential equations within the interval Δt. The occurrence of C indicates a step-wise change $\Delta_u x_l$, and thus when obtaining the first term of (VI) one has to determine the increment Δv corresponding to some value of the random quantity ν_l and to evaluate subsequently the mathematical expectation $\underset{\nu}{M}$ over the occurrences ν_l. The increments $\Delta v_{AC(\nu_1, \ldots, \nu_n)}$ are expanded in a Taylor's series up to the terms of the second order, neglecting the variations in the coordinates x_l, which are due to the fact that the coordinates x_l up to the step $\Delta_u x_l$ and later do not remain constant but vary along the trajectory of the system (1.2)-(1.3) when $\eta = \alpha_i$. The terms of the third order or higher in the expansion of the increment Δv_{AC} of the function v due to the step-wise changes $\Delta_u x_l$ in the coordinates are simply neglected because when $\lambda \to \infty$, $\sigma_l \to 0$, $\lambda\sigma_l^2$ = const [1] in the limit, these terms contribute infinitely small quantities of the order $o(\Delta t)$. Because $\Delta_u x_l = \mu_l \nu_l$ and $M(\nu_l) = 0$, $M(\nu_s \nu_l) = k_{ls}\sigma_l\sigma_s$ [1], therefore by dividing (VI) by Δt and by proceeding to the limit $\Delta t \to 0$ we obtain (1.6). To get (1.6) in a more general form (1.5) we proceed in the usual way from a finite sum to the integral.

LITERATURE CITED

1. N. N. Krasovskii and É. A. Lidskii, Analytic Design of Controllers in Systems with Random Attributes. Part I, Avtomatika i Telemekhanika, 22, No. 9 (1961).
2. B. V. Gnedenko, A Course in Theory of Probability, Gostekhizdat (1954).
3. I. Ya. Kats and N. N. Krasovskii, On Stability of Systems with Random Parameters. Prikl. Matem. i Mekh. 24, No. 5 (1960).
4. A. M. Lyapunov, The General Problem of Stability of Motion, Gostekhizdat (1950).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

ANALYTIC DESIGN OF REGULATORS (CONSTANT DISTURBANCES)

M. E. Salukvadze

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1279-1287, October, 1961
Original article submitted February 7, 1961

A linear system of automatic control which is optimal in a certain definite sense is considered. The equation for an optimum regulator is derived by applying a method of analytic design of regulators; the regulator may be put into effect by means of ordinary detective elements and regulating units.

1. Introduction

The dynamics of regulation of systems which are submitted to disturbing forces has continually drawn the attention of investigators. The trend to endow the self-regulating system with the ability to resist the disturbing forces has led to many concepts. There arose, in particular, the idea of controlling by load (and in the general case by disturbance) subsequently called the Poncelet-Chikolev principle; it became the subject matter of many investigations.

In order to clarify what is meant by these concepts it should be noted that any system possessing the property of asymptotic stability also has the property of resisting the disturbing forces provided the latter remain sufficiently small. This, however, does not suffice to be of much practical use.

Firstly, it is important to note that even though the disturbing forces are usually bounded, their sufficient smallness nevertheless cannot always be guaranteed. Secondly, should these disturbing forces remain sufficiently small, but still measurable (the usual situation in the case of Poncelet-Chikolev principle), or should they be given in the form of known functions of time, it is of considerable importance to reduce, as much as possible, the domain of feasible deviations of the system from its undisturbed motion. Poncelet, Chikolev and some more recent investigators have, in fact, had this particular goal in mind. It is our aim in the present paper to find the control rule in an analytic form and such that a measure which provides a description of the resistance of the system to the external forces will assume its extremal value.

The Poncelet-Chikolev principle, as well as the conditions governing its application to various types of control system, was investigated by V. S. Kulebakin, B. N. Petrov, A. Yu. Ishkinski, A. G. Ivakhnenko, G. M. Ulanov, and others.

In order to be concise the relevant literature is not cited here but can be found in [1, 2].

The author would like to express his gratitude to A. M. Letov for the formulation of the problem as well as for his aid in completing this work, and to N. N. Krasovskii for his most valuable comments.

2. Formulation of the Problem

An object under control is considered such that its disturbed motion is described in the neighborhood of the equilibrium position by the equations

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + m_k\xi + f_k(t) \quad (k = 1, \ldots, n). \tag{2.1}$$

Here η_k are the generalized coordinates of the object, ξ is the coordinate of the regulating device, $f_k(t)$ are the continuous and bounded time functions describing the external disturbing forces, and $b_{k\alpha}$ and m_k are the constant parameters of the object and of the regulating device respectively.

Let the equations (2.1) be defined in an open domain N and the coordinates η_k satisfy the obvious boundary conditions

$$\eta_1(0) = \eta_{10}, \eta_2(0) = \eta_{20}, \ldots, \eta_n(0) = \eta_{n0},$$
$$\eta_1(\infty) = \eta_2(\infty) = \ldots = \eta_n(\infty) = 0. \tag{2.2}$$

These signify that should a transient process occur in the domain N, it must then terminate when $t = \infty$ at the origin of the coordinate system. It is known that the conditions (2.2) can always be satisfied when the system (2.1) is closed and stable with $f_k(t) \equiv 0$ [3].

For the sake of simplicity we shall confine ourselves to continuous dwindling* functions $f_k(t)$, that is, to functions $f_k(t) = e^{-\delta t}\varphi_k(t)$, where $\varphi_k(t)$ is a bounded continuous function, and δ is a positive quantity, arbitrarily small [4-6].

Under these conditions we shall endeavor to find the functions $\xi, \eta_1, \ldots, \eta_n$ of the class C_1, related by the equations (2.1) and also minimizing the integral

$$I(\xi) = \int_0^\infty V dt \tag{2.3}$$

of the positive-definite quadratic form

$$V = \sum_{k=1}^n a_k \eta_k^2 + c\xi^2. \tag{2.4}$$

The integral for any $t > 0$ characterizes the disturbing effect of the external forces, and $1/I(\xi)$ therefore measures the resistance of the system to these forces. Other characteristics could, of course, be found to serve as another resistance measure. The measure presented here is closely related to the results obtained in [3].

Thus, the integral (2.3) has been selected to serve as the criterion of optimality of the system; it is a functional defined over the C_1 class of functions. The values of this integral characterize the integral square error weighted suitably by the constants a_k and c, the error being accumulated in the system throughout the duration of the transient process.**

The problem consists in determining the equation of the regulator which together with the equation (2.1) of the object of control represents the stable system with the highest measure of resistance to the given external disturbing forces $f_k(t)$.

If such an equation does exist, it must obviously correspond to the principle of combined control as described in [1, 2].

3. Solution

The problem under consideration is a variational problem of Lagrange. The required functions are $\xi, \eta_1, \ldots, \eta_n$ of the C_1 class such that they minimize the functional (2.3), and also satisfy the boundary conditions (2.2) and finally are connected by the relations

$$g_k \equiv \dot{\eta}_k - \sum_{\alpha=1}^n b_{k\alpha}\eta_\alpha - m_k\xi - f_k(t) = 0 \quad (k = 1, \ldots, n). \tag{3.1}$$

It is well known how to solve this kind of problem. The function

---

* A bounded function is said to be dwindling if it approaches zero when its argument increases without bounds.

** The choice of the parameters a_k and c is not discussed at this juncture.

$$H = V + \sum_{k=1}^{n} \lambda_k g_k \tag{3.2}$$

is formed, where λ_k are arbitrary Lagrange multipliers satisfying the equations

$$\begin{gathered} \dot{\lambda}_k = 2a_k\eta_k - \sum_{\alpha=1}^{n} b_{\alpha k}\lambda_\alpha \quad (k = 1, \ldots, n), \\ 0 = 2c\xi - \sum_{\alpha=1}^{n} m_\alpha \lambda_\alpha. \end{gathered} \tag{3.3}$$

If we add the equations (2.1) we obtain the complete system of the variational problem. The last (3.3) equation gives

$$\xi = \sum_{\alpha=1}^{n} \frac{m_\alpha}{2c} \lambda_\alpha. \tag{3.4}$$

Having put this ξ in (2.1) we obtain the equations of the variational problem in the form

$$\begin{gathered} \dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + m_k \sum_{\alpha=1}^{n} \frac{m_\alpha}{2c} \lambda_\alpha + f_k(t), \\ \dot{\lambda}_k = 2a_k\eta_k - \sum_{\alpha=1}^{n} b_{\alpha k}\lambda_\alpha. \end{gathered} \tag{3.5}$$

We solve this for η_k and λ_k. The characteristic determinant of the system is obviously

$$\begin{vmatrix} b_{11} - \mu, \ldots, & b_{1n}, & \frac{m_1^2}{2c} & , \ldots, & \frac{m_1 m_n}{2c} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n1}, \ldots, & b_{nn} - \mu, & \frac{m_n m_1}{2c} & , \ldots, & \frac{m_n^2}{2c} \\ 2a_1, \ldots, & 0, & -b_{11} - \mu, & \ldots, & -b_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, \ldots, & 2a_n, & -b_{1n}, & \ldots, & -b_{nn} - \mu \end{vmatrix}. \tag{3.6}$$

It is easy to show that if $\mu_1, \ldots, \mu_n$ are the simple roots of the equation

$$\Delta(\mu) = 0, \tag{3.7}$$

then the quantities $-\mu_1, \ldots, -\mu_n$ must also be simple roots [3]. We shall assume that $\mu_1, \ldots, \mu_n$, are, in fact, simple roots, numbered in such a way that the inequalities

$$\operatorname{Re} \mu_k < 0 \quad (k = 1, \ldots, n) \tag{3.8}$$

take place and thus the first n roots have negative real parts.

This can always be done in view of the previously mentioned property of the determinant (3.6). The general solution of the homogeneous system obtained from (3.5) will consist of a linear aggregate of exponential functions of the kind $c_k e^{\mu_k t}$, $c_{n+k} e^{-\mu_k t}$ $(k = 1, \ldots, n)$ and will contain 2n arbitrary constants $c_1, \ldots, c_n, c_{n+1}, \ldots, c_{2n}$:

$$\eta_\alpha = \sum_{k=1}^{n} A_k^\alpha c_k e^{\mu_k t} + \sum_{k=1}^{n} A_{n+k}^\alpha c_{n+k} e^{-\mu_k t},$$

$$\lambda_\alpha = \sum_{k=1}^{n} A_k^{n+\alpha} c_k e^{\mu_k t} + \sum_{k=1}^{n} A_{n+k}^{n+\alpha} c_{n+k} e^{-\mu_k t} \quad (\alpha = 1, \ldots, n), \tag{3.9}$$

where A_j^ν (ν, j = 1, . . . , 2n) are completely determined constants in terms of $b_{k\alpha}$, m_k, a_k, c.

In order to determine the general solution of (3.5) we apply the method of variation of parameters:

$$\sum_{k=1}^{n} A_k^\alpha \dot{c}_k e^{\mu_k t} + \sum_{k=1}^{n} A_{n+k}^\alpha \dot{c}_{n+k} e^{-\mu_k t} = f_\alpha(t),$$

$$\sum_{k=1}^{n} A_k^{n+\alpha} \dot{c}_k e^{\mu_k t} + \sum_{k=1}^{n} A_{n+k}^{n+\alpha} \dot{c}_{n+k} e^{-\mu_k t} = 0 \quad (\alpha = 1, \ldots, n). \tag{3.10}$$

With the assumptions made with respect to μ_k one is able to determine the coefficients c_k, c_{n+k} by means of the formulas which contain indefinite integrals only, that is,

$$c_k = \gamma_k + \sum_{i=1}^{n} D_i \int e^{-\mu_k t} f_i(t)\, dt,$$

$$c_{n+k} = \gamma_{n+k} + \sum_{i=1}^{n} D_{n+i} \int e^{\mu_k t} f_i(t)\, dt \quad (k = 1, \ldots, n). \tag{3.11}$$

The D_i, D_{n+i} (i = 1, . . . , n) can be found from the system (3.10) with the aid of Cramer's formulas; they will contain the coefficients A_j^ν (ν, j = 1, . . . , 2n). Consequently, the general solution of the nonhomogeneous system (3.5) is

$$\eta_\alpha = \sum_{k=1}^{n} A_k^\alpha \gamma_k e^{\mu_k t} + \sum_{k=1}^{n}\sum_{i=1}^{n} A_k^\alpha D_i e^{\mu_k t} \int e^{-\mu_k t} f_i(t)\, dt + $$
$$+ \sum_{k=1}^{n} A_{n+k}^\alpha \gamma_{n+k} e^{-\mu_k t} + \sum_{k=1}^{n}\sum_{i=1}^{n} A_{n+k}^\alpha D_{n+i} e^{-\mu_k t} \int e^{\mu_k t} f_i(t)\, dt,$$

$$\lambda_\alpha = \sum_{k=1}^{n} A_k^{n+\alpha} \gamma_k e^{\mu_k t} + \sum_{k=1}^{n}\sum_{i=1}^{n} A_k^{n+\alpha} D_i e^{\mu_k t} \int e^{-\mu_k t} f_i(t)\, dt + \tag{3.12}$$
$$+ \sum_{k=1}^{n} A_{n+k}^{n+\alpha} \gamma_{n+k} e^{-\mu_k t} + \sum_{k=1}^{n}\sum_{i=1}^{n} A_{n+k}^{n+\alpha} D_{n+i} e^{-\mu_k t} \int e^{\mu_k t} f_i(t)\, dt \quad (\alpha = 1, \ldots, n).$$

In accordance with the condition (2.2) at infinity we obtain

$$\gamma_{n+k} = 0 \quad (k = 1, \ldots, n). \tag{3.13}$$

In view of the assumptions made with regard to μ_k the constants γ_k can be chosen so that they agree with the initial conditions, the choice however not being unique.

Taking into account (3.13) we obtain from the first n equations of the system (3.12) a relation in the form

$$\left(\gamma_k + \sum_{i=1}^{n} D_i \int e^{-\mu_k t} f_i(t)\, dt\right) e^{\mu_k t}$$

and we substitute it into the remaining (3.12) equations.

It is assumed that it is possible to perform this operation. As a result we obtain

$$\lambda_\alpha = \sum_{k=1}^{n} B_k^\alpha \eta_k + \sum_{k=1}^{n} B_{n+k}^\alpha \sum_{i=1}^{n} D_i e^{-\mu_k t} \int e^{\mu_k t} f_i(t)\, dt \quad (\alpha = 1, \ldots, n). \tag{3.14}$$

The coefficients B_k^α, B_{n+k}^α ($k, \alpha = 1, \ldots, n$) are completely determined, and the D_i ($i = 1, \ldots, n$) are finally expressed in terms of the parameters $b_{k\alpha}$ and m_k of the system, and in terms of the weighting coefficients a_k and $\underline{c}$ of the functional (2.3).

As a result of this substitution of (3.14) into (3.4) the required equation of the regulator is obtained satisfying the Poncelet-Chikolev principle:

$$\xi = \sum_{\alpha=1}^{n} p_\alpha \eta_\alpha + \sum_{\alpha=1}^{n} p_{n+\alpha} \sum_{i=1}^{n} D_i e^{-\mu_\alpha t} \int e^{\mu_\alpha t} f_i(t)\, dt. \tag{3.15}$$

This equation together with the original equations (2.1) describes a stable system of automatic control when $f_k(t) \equiv 0$ ($k = 1, \ldots, n$). This is so because the homogeneous system of the found optimum system has a linear aggregate of exponential functions $c_k e^{\mu_k t}$ ($k = 1, \ldots, n$) as its general solution where μ_k have the property (3.8).

The obtained results can easily be generalized for the case when the control is effected by means of n regulating devices; in such a case the equations of the variational problem are

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha} \eta_\alpha + \sum_{\alpha=1}^{n} m_{k\alpha} \xi + f_k(t),$$

$$\dot{\lambda}_k = 2a_k \eta_k - \sum_{\alpha=1}^{n} b_{\alpha k} \lambda_\alpha, \tag{3.16}$$

$$0 = 2c^{(k)} \xi_k - \sum_{\alpha=1}^{n} m_{\alpha k} \lambda_\alpha \quad (k = 1, \ldots, n).$$

By solving them in a similar manner we obtain the equations of the regulating devices in the analytic form

$$\xi_k = \sum_{\alpha=1}^{n} p_\alpha^{(k)} \eta_\alpha + \sum_{\alpha=1}^{n} p_{n+\alpha}^{(k)} \sum_{i=1}^{n} \bar{D}_i e^{-\mu_\alpha t} \int e^{\mu_\alpha t} f_i(t)\, dt \quad (k = 1, \ldots, n). \tag{3.17}$$

The coefficients $p_\alpha^{(k)}$, $p_{n+\alpha}^{(k)}$, $\bar{D}_i$ ($\alpha, k, i = 1, \ldots, n$) are evaluated by the step-by-step method and are finally expressed in terms of the system parameters $b_{k\alpha}$, $m_{k\alpha}$, and of the weighting coefficients a_k, $c^{(k)}$ of the minimized functional.

As can be seen from the obtained equations (3.15) and (3.17), the equations of the optimum regulator contain the external disturbances $f_k(t)$ under the indefinite integral sign. As previously mentioned the integration constants must vanish [see (3.13)]. As noted by N. N. Krasovskii the latter can always be attained provided that the functions $f_k(t)$ are known over the whole interval $t(0, \infty)$. In this connection it is of interest to consider two cases:

a) The functions $f_k(t)$ are known. The indefinite integrals of the known functions can then be found, and the optimum regulator can be designed in accordance with (3.17). Such a regulating device is considered later in the sections 5 and 6.

b) The functions $f_k(t)$ are not known but the values of the functions $f_k(t)$, $f_k'(t)$, $f_k''(t)$, ... can be measured at any time instant.

In this case, after successive integrations by parts, the expression (3.15) becomes *

$$\xi = \sum_{\alpha=1}^{n} p_\alpha \eta_\alpha + \sum_{\alpha=1}^{n} p_{n+\alpha} \sum_{i=1}^{n} D_i \sum_{r} \frac{(-1)^r}{\mu_\alpha^{r+1}} f_i^{(r)}(t) \qquad (r = 0, 1, 2, \ldots). \tag{3.18}$$

The character of the transient process depends on how the index r is chosen. The bigger it is, the nearer is the transient process in the system to the optimal one in the sense of (2.3).

The choice of r should be governed by the specific possibilities inherent in the design of the given automatic control system.

4. Verification of Boundary Conditions

When the equation of the regulator is (3.15) the coordinates η_k ($k = 1, \ldots, n$) of the controlled object vary in accordance with the formulas (3.12) and (3.13).

Of course it can be verified straightaway that the initial conditions are fulfilled when the μ_k are simple roots. We shall now verify the conditions at infinity. It is sufficient to show that the expression

$$e^{-\mu t} \int e^{\mu t} f_k(t)\, dt \tag{4.1}$$

is a dwindling function of time for any $k = 1, \ldots, n$.

When formulating the problem it was stated that the functions $f_k(t)$ ($k = 1, \ldots, n$) are dwindling functions, that is,

$$f_k(t) = e^{-\delta t} \varphi_k(t) \qquad (k = 1, \ldots, n),$$

where $|\varphi_k(t)| \leq L$, L is a positive quantity, and δ is positive and sufficiently small. Then

$$\left| e^{-\mu t} \int e^{\mu t} f_k(t)\, dt \right| \leqslant L \left| e^{-\mu t} \int e^{(\mu - \delta) t} dt \right| = \frac{L}{|\mu - \delta|} e^{-\delta t}. \tag{4.2}$$

Consequently, the expression (4.1) represents a dwindling function.

5. Example

A system of the first order whose variational problem equations are

$$\dot{\eta} = b\eta + m\xi + f(t),\ \dot{\lambda} = 2a\eta - b\lambda,\ 0 = 2c\xi - m\lambda \tag{5.1}$$

will be examined as an example.

The corresponding characteristic equation is

$$\mu^2 - b^2 - \frac{m^2 a}{c} = 0,\ \mu_1, \mu_2 = \mp \sqrt{b^2 + \frac{m^2 a}{c}} \tag{5.2}$$

and the regulator equation is

$$\xi = \frac{\mu_1 - b}{m} \eta - \frac{\mu_1 - b}{m} e^{-\mu_1 t} \int e^{-\mu_1 t} f(t) dt. \tag{5.3}$$

* $f_1^{(r)}(t)$ denotes the rth derivative with respect to t of $f_1(t)$. Of course, the series $\sum_{r=0}^{\infty} \frac{(-1)^r}{\mu_\alpha^{r+1}} f_i^{(r)}(t)$ is convergent for any $t \geq 0$.

In the case of $f(t) \equiv 0$ the equation (5.3) is identical with the one obtained in [3].

This example can also serve to demonstrate a different approach to the solution: use is made of a fundamental concept in the analytic theory of differential equations; that is,that, with some restrictions on f(t), the latter may be considered as a particular solution of another differential equation [7].

For the sake of simplicity, it is assumed that $f(t) = \zeta$ satisfies a very simple equation of the form

$$F(\dot{\zeta}, \zeta, t) = 0, \quad \zeta(0) = f(0), \tag{5.4}$$

which can now be treated as an equation of an element connected to the object under control.

As, by assumption, the function f(t) is a dwindling function, we have $\zeta(\infty) = 0$. Then (5.1) is replaced by the following equations of the variational problem:

$$\begin{gathered} \dot{\eta} = b\eta + \zeta + m\zeta, \quad F(\dot{\zeta}, \zeta, t) = 0, \quad \dot{\lambda}_1 = 2a\eta - b\lambda_1, \\ \frac{d}{dt}\left(\lambda_2 \frac{\partial F}{\partial \dot{\zeta}}\right) = -\lambda_1 + \lambda_2 \frac{\partial F}{\partial \zeta}, \quad 0 = 2c\xi - m\lambda_1. \end{gathered} \tag{5.5}$$

The boundary conditions have already been discussed. The first, third and the fifth equations can each be integrated independently of one another. Under these conditions the equations for the optimal system can now be written in the form

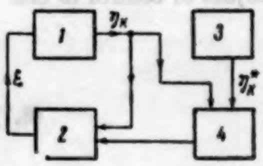
$$\begin{gathered} \dot{\eta} = b\eta + \zeta + m\xi, \quad F(\dot{\zeta}, \zeta, t) = 0, \\ \zeta = \frac{\mu_1 - b}{m}\eta - \frac{\mu_1 - b}{m} e^{\mu_2 t} \int e^{-\mu_2 t} \zeta dt. \end{gathered} \tag{5.6}$$

The last formula in the regulator equation may be looked upon as a controlling signal indicating the deviations from the equilibrium position of the adjoint element ξ, which can be used to increase the measure of resistance of the controlled object to the disturbing forces.

6. A Simple Self-Adaptive System

Many tvpes of self-adaptive systems can be found in literature [8]. We shall describe a simple case shown diagramatically in the figure below [16].

The object-regulator part of the scheme operating in the ordinary manner, without adaptation, shall first be considered. If the characteristics of the object are known and are known to be stable, a regulator can be designed which will provide the system with an entirely satisfactory measure of control. However, it often occurs that the parameters of the object are not stable and vary in a haphazard way. In the simple case, for instance, when the impressed motion of the object can be described by the following equations:



1) Controlled object; 2) regulator; 3) analogue of the true object with control; 4) analyzer.

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + m_k\xi \quad (k = 1, \ldots, n), \tag{6.1}$$

where η_k are the generalized coordinates of the object, ξ the coordinate of the controlling unit, and the parameters $b_{k\alpha}$ of the object and m_k of the controlling unit may vaty within a given bounded set D of real numbers [9]. The parameters $b_{k\alpha}$ and m_k vary in a random manner but in such a way that they remain constant within a sufficiently long time interval. The probabilistic characteristics of this process remain unknown.

For any fixed combination of $b_{k\alpha}, m_k \in D$ the matching parameters $p_1, p_2, \ldots, p_n$ of the regulator can be found such that the system as a whole has the required characteristics. The latter, however, shall not be maintained if the parameters $b_{k\alpha}$ and m_k vary in the set D. Under these circumstances there arises the problem of adapting the system to the variations of $b_{k\alpha}$ and m_k, that is,of having a new set of matching parameters of the regulator such that the whole system possesses again the required characteristics.

The adaptation process depends primarily on how the characteristics which the system should possess are formulated. This may be done as follows. We assume that a standard [9] of the object is defined in the set D, and that it is introduced into the system in the form of an electronic analogue corresponding to the equations

$$\dot{\eta}_k^* = \sum_{\alpha=1}^{n} b_{k\alpha}^* \eta_\alpha^* \quad (k = 1, \ldots, n). \tag{6.2}$$

We also assume that the equalities

$$\eta_k(0) = \eta_k^*(0) \quad (k = 1, \ldots, n), \tag{6.3}$$

are true for the analogue, that is, that the motions of the object and of its analogue start from the same initial deviations of the regulated coordinates. It seems feasible to suppose that the analogue of the standard satisfies the requirements of the Lyapunov asymptotic stability, and also that the transient process η_k^* satisfies the previously formulated requirements of the standard. An analyzer is also connected to the analogue which compares the transient processes η_k^* of the standard and η_k of the real object of control. The comparison can be carried out in several ways. Let us consider the case when the analyzer produces such an equation of the regulator that the least value of the integral

$$I = \int_0^\infty \Big[\sum_{k=1}^{n} a_k (\eta_k - \eta_k^*)^2 + c\xi^2 \Big] dt \tag{6.4}$$

is attained.

The main difficulty lies in determining such an equation. We now assume that in some way we have gotten to know the values of $b_{k\alpha}$ and m_k.

The new variables θ_k are introduced, defined by the formulas

$$\theta_k = \eta_k - \eta_k^* \quad (k = 1, \ldots, n). \tag{6.5}$$

With the aid of the formulas (6.1), (6.2) and (6.5) one can find the equations describing θ_k:

$$\dot{\theta}_k = \sum_{\alpha=1}^{n} b_{k\alpha} \theta_\alpha + m_k \xi + \sum_{\alpha=1}^{n} (b_{k\alpha} - b_{k\alpha}^*) \eta_\alpha^* \quad (k = 1, \ldots, n). \tag{6.6}$$

The equation (6.6) differs from the equation (6.1) by the presence of the last component. Consequently, in finding a regulator which minimizes the value of the integral (6.4), one has to submit the object of control to constant disturbances equal to

$$f_k(t) = \sum_{\alpha=1}^{n} (b_{k\alpha} - b_{k\alpha}^*) \eta_\alpha^* \quad (k = 1, \ldots, n). \tag{6.7}$$

The functions $f_k(t)$ must be dwindling in view of the fact that the functions η^*_α, which are the solutions of the system (6.2), are stable by our assumption. Consequently, the equations of the required regulator can be written in the form (3.15), where $f_k(t)$ are determined by (6.7).

The parameters $b_{k\alpha}$ and m_k may be determined experimentally. To this end one can avail oneself of the methods developed in [10-15].

The system is in such case submitted to tests, and the information obtained yields the values of $b_{k\alpha}$ and m_k. After each process a correction should be introduced into the equation of the regulator.

This process can be made to be automatic by means of a computer working out the values of the parameters with the aid of an agreed algorithm for evaluation from experimental data of the parameters of the object.

LITERATURE CITED

1. G. M. Ulanov, Regulation by Disturbing Force, Gosénergoizdat (1960).
2. A. G. Ivakhnenko, Technical Cybernetics, Gostekhizdat, Ukr. SSR (1959).
3. A. M. Letov, Analytic Design of Regulator, I, II, and III, Avtomatika i Telemekhanika, 21, Nos. 4, 5, 6 (1960).
4. N. G. Chetaev, Stability of Motion, Gostekhizdat (1955).
5. I. G. Malkin, Theory of Stability of Motion, Gostekhizdat (1952).
6. N. N. Krasovskii, Some Problems of the Theory of Stability of Motion, Fizmatgiz (1959).
7. V. V. Golubev, Lectures on Analytic Theory of Differential Equations, Gostekhizdat (1950).
8. A. A. Fel'dbaum, Computers in Automatic Systems, Fizmatgiz (1959).
9. O. Kh. Shukk, Self-Adaptive Autopilots, Proc. of the First International Conference of Automatic Control, USSR Acad. of Sci. Press (1961).
10. N. Wiener, Nonlinear Problems in Random Processes, Massachusetts (1960).
11. A. Bose, A Theory of Nonlinear Systems, Massachusetts Institute of Technology, Research Laboratory of Electronics. Technical Report No. 309 (May 15, 1956).
12. Braun, Mishkin, and Traksel, Approximate Determination of Dynamic Characteristics of a Controlled Process in Self-Adaptive Systems, Proc. of the First International Conference on Automatic Control, USSR Acad. of Sci. Press (1961).
13. T. Kitamori, Orthogonal Functions Applied to the Determination of the Dynamic Characteristics of Objects and the Design of Self-Optimizing Systems of Automatic Control, Proc. of the First International Conference on Automatic Control, USSR Acad. of Sci. Press (1961).
14. G. I. Monastyrshin, The Processing of Experimental Frequency Characteristics, Avtomatika i Telemekhanika, 21, No. 3 (1960).
15. G. Smith, Synthesis of a Self-Adaptive Autopilot for a Large Elastic Booster, IRE International Convention Record, Part 4–Automatic Control; Information Theory (March 21-24, 1960).
16. R. E. Kalman, Design of Self-Optimizing Control System, 57-IRD-12.

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

THE MAXIMUM PRINCIPLE FOR OPTIMUM SYSTEMS WITH DISTRIBUTED PARAMETERS

A. G. Butkovskii

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1288-1301, October, 1961
Original article submitted May 18, 1961

The problem of optimal control is considered for systems with distributed parameters, the motion of which is described by nonlinear integral equations. A theorem formulated as a maximum principle is introduced which can be used to determine the extremals for any general variational problems, where it is desired to minimize an arbitrary functional with supplementary boundaries.

The essential and most general results in the theory of optimal automatic control systems [1, 2] have been obtained for objects of control with lumped parameters, the motion of which is described by a system of ordinary differential equations

$$\dot{x} = f(x, u), \tag{1}$$

where the matrix function

$$f(x, u) = \left\| \begin{matrix} f^1(x, u) \\ \vdots \\ f^n(x, u) \end{matrix} \right\| \tag{2}$$

of the variables $\underline{x}$ and $\underline{u}$ is defined on the direct product

$$(x, u) \in X^n\Omega, \quad x \in X^n, \quad u \in \Omega. \tag{3}$$

Here X^n is an n-dimensional phase space, and Ω an arbitrary, topological, Hausdorff space of the permissible values of the control parameter $\underline{u}$. The optimal control problem is that of choosing from the class of permissible controls (for example, in the class of measurable, bounded functions, or in the class of piecewise continuous functions, with values from Ω) a function $u(t)$ ($t_0 \leq t \leq t_1$), for which the control trajectory $x(t)$ for Eq. (1) traverses a path from the point $x_0 = x_0(t_0)$ to some manifold M_s. The dimension $\underline{s}$ of this manifold should not exceed $n-1$, and the integral

$$x^0 = \int_{t_0}^{t_1} f^0(x, u)\, dt \tag{4}$$

along the path should be a minimum, where $f^0(x, u)$ is a known function of its arguments, defined on the same set as the function $f(x, u)$.

In practice, however, objects of control with distributed parameters occur much more often than objects with lumped parameters or objects that can be reduced to the case of lumped parameters. It is therefore of great importance to obtain an optimal control theory for systems with such distributions. To this class belong, for example, continuous production lines, and also a whole series of objects involved in production.

In many cases, a design problem consists of the choice of a certain definite optimal distribution of parameters of the assembly to be constructed. For such systems, a control system must be constructed which will give the best possible performance in a definite sense, when certain conditions are fulfilled. When it is a question of design, then the optimal distribution must be determined of those parameters that are at the choice of the designer.

We will consider an object of control, the state of which is characterized by a one-column matrix function

$$Q = Q(P) = \begin{Vmatrix} Q^1(P) \\ \vdots \\ Q^n(P) \end{Vmatrix}, \tag{5}$$

where P is a point in a certain m-dimensional region D of the Euclidian space $E(y_1, \ldots, y_m)$.

The matrix function $Q = Q(P)$ will be called the state of the system.

Objects having parameters distributed in the space can be affected by certain controlling actions, also distributed in space and time. In other words, the controlling action is described by the vector function

$$u = u(S), \tag{6}$$

where the point $S \in D$.

Each component of this control vector has the form

$$u_{\alpha\beta} = u_{\alpha\beta}(y_1, \ldots, y_\alpha) = u_{\alpha\beta}(S_\alpha) \quad (\alpha = 1, 2, \ldots, m;\ \beta = 1, 2, \ldots, \rho_\alpha), \tag{7}$$

where ρ_α is a positive integer dependent on α.

Thus the vector $u = u(S)$ consists of

$$r = \sum_{\alpha=1}^{m} \rho_\alpha \tag{8}$$

components. Here the functions $u_{\alpha\beta}(S_\alpha)$ are defined only on the set of points $S_\alpha = (y_1, \ldots, y_\alpha) \in N_\alpha \subset D$ $(\alpha = 1, 2, \ldots, m)$, where N_α is some manifold of dimension α. We will assume that the functions $u_{\alpha\beta}$ are measurable, bounded, and square-integrable functions of S_α, and that their values are in some admissible set Ω, which can, in general, be any topological Hausdorff space. In particular, an important case is when Ω is a closed region of some r-dimensional space E_Ω.

We call each vector function $u = u(S)$ taking values in the space Ω a control.

The process of controlling the object consists of finding a state $Q = Q(P)$ to correspond to each control $u = u(S)$ according to a definite law. The law defining such a correspondence can be specified by a certain operation A:

$$Q = Au. \tag{9}$$

In other cases, the law of correspondence between $\underline{u}$ and Q can also be given by more complicated functional equations.

A wide class of objects of control can be described by nonlinear integral equations of the type

$$Q(P) = \int_D K\big(P, S, Q(S), u(S)\big)\, dS, \tag{10}$$

where Q(P) and u(S) are defined by (5)-(7), and $K = K(P, S, Q, u)$ is a one-column matrix function of the four vector variables:

$$K(P, S, Q, u) = \left\| \begin{matrix} K^1(P, S, Q, u) \\ \cdot \\ \cdot \\ \cdot \\ K^n(P, S, Q, u) \end{matrix} \right\| . \tag{11}$$

The functions $K^i(P, S, Q, u)$ $(i = 1, 2, \ldots, n)$ belong to the class L_2, and have the continuous partial derivatives

$$\frac{\partial K^i}{\partial Q^j} = \frac{\partial K^i(P, T, Q, u)}{\partial Q^j} \qquad (j = 1, 2, \ldots, n). \tag{12}$$

We now introduce the square matrix

$$\frac{\partial K}{\partial Q} = \left\| \frac{\partial K^i}{\partial Q^j} \right\| \qquad (i, j = 1, 2, \ldots, n). \tag{13}$$

The superscript $\underline{i}$, here and in the sequel, indicates the row number, while $\underline{j}$ gives the column of the matrix.

In this case, the problem of optimal control with distributed parameters can be stated as follows.

On a set of states $Q = Q(P)$ and controls $u = u(S)$, related by the integral equation (10), let there be defined $\underline{q}$ functionals having continuous gradients:

$$I^i = I^i(Q(P)) \quad (i = 0, 1, 2, \ldots, l), \tag{14}$$

$$I^i = I^i(Q(P), u(P)) = \Phi^i(z) \qquad (i = l + 1, \ldots, q), \tag{15}$$

where

$$z = \left\| \begin{matrix} z^0 \\ \cdot \\ \cdot \\ z^k \end{matrix} \right\| = \left\| \begin{matrix} \int_D F^0(S, Q(S), u(S))\, dS \\ \cdot \\ \cdot \\ \int_D F^k(S, Q(S), u(S))\, dS \end{matrix} \right\| = \int_D F(S, Q(S), u(S))\, dS, \tag{16}$$

$$F(S, Q, u) = \left\| \begin{matrix} F^0(S, Q, u) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F^k(S, Q, u) \end{matrix} \right\| . \tag{17}$$

The functions $\Phi^i(z)$ $(i = l + 1, \ldots, q)$ and $F^i(S, Q, u)$ $(i = 0, 1, \ldots, k)$ are continuous, and have continuous partial derivatives with respect to the components of the matrices $\underline{z}$ and Q respectively.

Here the functionals I^i $(i = l + 1, \ldots, q)$ can clearly depend on the control $\underline{u}$.

We note that the functionals I^i $(i = 0, 1, \ldots, q)$ can also be determined when the point P does not necessarily run through the whole region D, but only through a subset (possibly of lower dimension) of this region.

Among the permissible controls, we wish to find a control u = u(P) (u ∈ Ω), for which

$$I^i = 0 \quad (i = 0, 1, \ldots, p-1,\ p+1, \ldots, q), \tag{18}$$

with the functional I^p taking its least possible value. Here p is the index of the functional, $0 \le p \le q$.

A control u = u(P) yielding the solution of this problem we call an optimal control.

We introduce the following rectangular matrices:

$$\frac{\partial \Phi}{\partial z} = \left\| \frac{\partial \Phi^i}{\partial z^j} \right\| \quad (i = 0, 1, \ldots, l;\ j = 0, 1, \ldots, k), \tag{19}$$

$$\frac{\partial F}{\partial Q} = \left\| \frac{\partial F^i}{\partial Q^j} \right\| \quad (i = 0, 1, \ldots, k;\ j = 1, 2, \ldots, n), \tag{20}$$

$$\operatorname{grad} I = \| \operatorname{grad}_j I^i \| \quad (i = l+1, \ldots, q;\ j = 1, 2, \ldots, n). \tag{21}$$

Here $\operatorname{grad}_j I^i$ denotes the jth component of the vector grad I^i in the coordinates Q^j.

The solution of the above-formulated problem can be based on the following theorem, the proof of which is given in Appendix 1.

Theorem. Let u = u(S) (u ∈ Ω) be a permissible control, such that, in view of the equation (10), the conditions

$$I^i = 0 \quad (i = 0, 1, \ldots, p-1,\ p+1, \ldots, q)$$

are satisfied, and let the matrix function $M(P, R) = \|M_{ij}(P, R)\|$ (i, j = 1, 2, . . . , n) satisfy the integral equation (linear in M(P, R))

$$M(P, R) + \frac{\partial K(P, R, Q(R), u(R))}{\partial Q} = \int_D M(P, S) \frac{\partial K(S, R, Q(R), u(R))}{\partial Q} dS =$$

$$= \int_D \frac{\partial K(P, S, Q(S), u(S))}{\partial Q} M(S, R)\, dS. \tag{22}$$

Then, in order that this control u = u(S) be optimal, it is necessary that there exist non-zero, one-column, numerical matrices

$$a = \| c_0, c_1, \ldots, c_l \|, \quad b = \| c_{l+1}, \ldots, c_q \|, \tag{23}$$

with $c_p = -1$, such that for almost all fixed values of the argument S ∈ D the function*

$$\Pi(S, u) = a \frac{\partial \Phi\left(\int_D F(P, Q(P), u(P))\, dP\right)}{\partial z} \times$$

$$\times \left(\frac{\partial F(P, Q(P), u(P))}{\partial Q} K(P, S, Q(S), u) - \int_D M(P, R) K(R, S, Q(S), u)\, dR \right) +$$

$$+ a \frac{\partial \Phi\left(\int_D F(P, Q(P), u(P))\, dP\right)}{\partial z} F(S, Q(S), u) +$$

$$+ b\left(\operatorname{grad} I(Q(P)), K(P, S, Q(S), u) - \int_D M(P, R) K(R, S, Q(S) u)\, dR\right) \tag{24}$$

of the variable u ∈ Ω attains its maximum, i.e., for almost all S ∈ D, the relation

$$\Pi(S, u) = H(S) \tag{25}$$

holds.

* Two expressions in brackets, separated by commas, denote a scalar product.

The function H = H(S) is defined to be the upper bound of the values of the function $\Pi(S, u)$ for $u \in \Omega$ and a fixed value of S:

$$H(S) = \sup_{u \in \Omega} \Pi(S, u). \tag{26}$$

The above theorem can be used to obtain a system of equations which is satisfied by an optimal control.

This result is more general than the result obtained in [4].

The following problem, in fact, is posed in [4]. The object of control is described by the integral relation

$$Q^i = Q^i(t) = \int_{t_0}^{t_1} K^i\left((t, \tau, u_1(\tau), \ldots, u_r(\tau)\right) d\tau. \tag{27}$$

It is desired to find an optimal control $u = u(t) = (u_1(t), \ldots, u_r(t))$, $t_0 \leq t \leq t_1$, $u \in \Omega$, where Ω is a closed region of r-dimensional Euclidian space, such that the functional

$$Q^0 = \int_{t_0}^{t_1} F^0(\tau, Q(\tau), u(\tau))\, d\tau \tag{28}$$

takes its minimum value for the condition that the point $Q(t_1) \in M$, where M is some manifold of arbitrary dimension s not exceeding $n-1$ in the n-dimensional Euclidian space $E(Q^1, \ldots, Q^n)$.

We will show that Theorems 1 and 2 of [4] follow from the above-stated theorem.

Let the equations of the manifold M have the form

$$\varphi^i = \varphi^i(Q^1, \ldots, Q^n) = \varphi^i(Q) \quad (i = 1, 2, \ldots, n-s), \tag{29}$$

where the functions $\varphi^i(Q)$ are continuous, and have continuous partial derivatives. We introduce the matrices

$$\frac{\partial \varphi}{\partial Q} = \left\| \frac{\partial \varphi^i}{\partial Q^j} \right\| \quad (i = 1, 2, \ldots, n-s,\ j = 1, 2, \ldots, n), \tag{30}$$

$$K(t, \tau, u) = \| K^i(t, \tau, u) \| \quad (i = 1, 2, \ldots, n). \tag{31}$$

We also define I^0 and $\Phi^0(z) \equiv z^0$:

$$I^0 = Q^0 = \int_{t_0}^{t_1} F(\tau, Q(\tau), u(\tau))\, d\tau = \Phi^0(z) \equiv \int_{t_0}^{t_1} F^0(\tau, Q(\tau), u(\tau))\, d\tau, \tag{32}$$

$$\Phi^i(z) \equiv \varphi^i(z) \quad (i = 1, 2, \ldots, n-s). \tag{33}$$

In this notation, the last problem can be stated in the following way. Let the region D be an interval $[t_0, t_1]$, and let the point $S = \tau \in [t_0, t_1]$. We wish to find an optimal control $u = u(\tau)$, $t_0 \leq \tau \leq t_1$, $u \in \Omega$, such that the functional

$$I^0 = \Phi^0(z) \equiv \int_{t_0}^{t_1} F^0(\tau, Q(\tau), u(\tau), u(\tau))\, d\tau$$

has a maximum for the condition

$$I^i = \Phi^i(z) = \Phi^i(Q(t)) = \Phi^i\left(\int_{t_0}^{t_1} K^i(t_1, \tau, u(\tau))\, d\tau\right) \quad (i = 1, 2, \ldots, n). \tag{34}$$

Here the functions $F(\tau, Q, u)$ and $K^i(t_1, \tau, u)$ play the role of the functions $F^i(S, Q, u)$ in the equation (16).

Before we apply the theorem, we note that, since the matrix function $K(t_1, \tau, u)$ in this case is independent of the variable Q, the only matrix function satisfying the equations (22) will be the function

$$(M(P, R)) \equiv 0, \quad P \in D, R \in D. \tag{35}$$

Applying the theorem stated above, we find that for the existence of an optimal control $u = u(\tau)$, it is necessary that there exist a non-zero, one-row, numerical matrix

$$a = \| c_0, c_1, \ldots, c_{n-s} \|, \tag{36}$$

with $c_0 = -1$, such that, for almost all fixed values of the argument τ the function $\Pi(\tau, u)$, having the expanded form

$$\Pi(\tau, u) = c_0 \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial F(t, Q(t), u(t))}{\partial Q^i} K^i(t, \tau, u)\, dt + \\ + c_0 F(\tau, Q(\tau), u) + \sum_{j=1}^{n} \sum_{i=1}^{n-s} c_i \frac{\partial \varphi^i(Q(t_1))}{\partial Q^j} K^j(t_1, \tau, u), \tag{37}$$

gives the maximum for the variable $u \in \Omega$.

We now set

$$\xi_j = \sum_{i=1}^{n-s} c_i \frac{\partial \varphi^i(Q(t_1))}{\partial Q^j} \quad (j = 1, 2, \ldots, n). \tag{38}$$

The last equality means that the vector $\xi = (\xi_1, \ldots, \xi_n)$ must be orthogonal to the manifold M at the point $Q(t_1) \in M$, i.e., at the point $Q(t_1)$ the transversality condition is satisfied.

The conditions (36)-(38) obtained above form the results of [4] stated in the theorems 1 and 2.

We now show how to apply the theorem obtained above to obtain the conditions for a maximum and for transversality, obtained previously in [1], for the problems (1)-(4) formulated in the present article.

We note, first of all, that the following lemma, proved in Appendix 2, is valid.

Lemma. Let the matrix functions $\mu = \mu(t, \tau)$ and $\nu = \nu(t, \tau)$, t and $\tau \in [t_0, t_1)$ be related by the equality

$$\mu(t, \tau) + \nu(t, \nu) = \int_{t_0}^{t_1} \mu(t, \theta)\, \nu(\theta, \tau)\, d\theta. \tag{39}$$

If the function $\nu = \nu(t, \tau)$ is defined to be

$$\nu = \nu(t, \tau) = \begin{cases} \bar{\nu}(\tau) & \text{for } t_1 \geqslant t \geqslant \tau \geqslant t_0, \\ 0 & \text{for } t_1 \geqslant \tau \geqslant t \geqslant t_0, \end{cases} \tag{40}$$

where $\bar{\nu}(\tau)$ is square-integrable in $[t_0, t_1)$, then the function $\mu(t, \tau)$ satisfies, for $t_1 \geq t \geq \tau \geq t_0$, the equation

$$\mu(t, \tau) + \bar{\nu}(\tau) = \int_{\tau}^{t_1} \mu(t, \theta)\, d\theta\, \bar{\nu}(\nu\tau) \tag{41}$$

and for $t_1 \geq \tau > t \geq t_0$

$$\mu(t, \tau) \equiv 0. \tag{42}$$

We now turn to the derivation of the conditions for a maximum and for transversality, obtained in [1].

Let the equations of the manifold M in the space X^n be

$$\varphi^i = \varphi^i(x) \quad (i = 1, 2, \ldots, n-s). \tag{43}$$

The region D is the interval $[t_0, t_1]$, and the points P and S are given by

$$P = t \in [t_0, t_1] \text{ and } S = \tau \in [t_0, t_1]. \tag{44}$$

We write the differential equation (1) and the equality (4) in the form of an integral equation

$$\bar{x}(t) = \int_{t_0}^{t_1} \tilde{f}(t, \tau, \bar{x}(\tau), u(\tau))\, d\tau, \tag{45}$$

where

$$\tilde{f}(t, \tau, \bar{x}(\tau), u(\tau)) = \begin{cases} f^i(x(\tau), u(\tau)) + \dfrac{x_0^i}{t_1 - t_0} & \text{for } t_1 \geqslant t \geqslant \tau \geqslant t_0 \\ 0 & \text{for } t_1 \geqslant \tau > t \geqslant t_0 \end{cases} \tag{46}$$

$$(i = 0, 1, 2, \ldots, n; \quad x_0^0 = 0).$$

If we now set

$$Q(t) = \begin{Vmatrix} x^0(t) \\ \vdots \\ x^n(t) \end{Vmatrix} = \bar{x}(t), \tag{47}$$

$$K(t, \tau, Q, u) = \|\tilde{f}^i(t, \tau, \bar{x}, u)\| = \tilde{f}(t, \tau, \bar{x}, u) \quad (i = 0, 1, \ldots, n), \tag{48}$$

$$I^0 = \varphi^0(z) = \int_{t_0}^{t_1} f^0(x(\tau), u(\tau))\, d\tau = \Phi^0(z) \equiv \int_{t_0}^{t_1} F^0(\tau, Q(\tau), u(\tau))\, d\tau, \tag{49}$$

$$\varphi^0(z) \equiv z^0, \tag{50}$$

$$I^i = \varphi^i\Big(\int_{t_0}^{t_1} f(x(\tau), u(\tau))\, d\tau\Big) = \Phi^i\Big(\int_{t_0}^{t_1} F(\tau, Q(\tau), u(\tau))\, d\tau\Big)$$

$$(i = 1, 2, \ldots, n-s), \tag{51}$$

then with this notation the function $\Pi(\tau, u)$ defined in (24) will become

$$\Pi(\tau, u) = a\, \frac{\partial\varphi\big(\int_{t_0}^{t_1} f(x(t), u(t))\, dt\big)}{\partial x} \int_{t_0}^{t_1} \frac{\partial f(x(t), u(t))}{\partial x} \Big(\tilde{f}(t, \tau, \bar{x}(\tau), u) -$$

$$- \int_{t_0}^{t_1} M(t, \theta)\, \tilde{f}(\theta, \tau, x(\tau), u)\, d\theta\Big)\, dt\Big) + a\, \frac{\partial\varphi\big(\int_{t_0}^{t_1} f(x(t), u(t))\, dt\big)}{\partial x} \tilde{f}(t_1, \tau, \bar{x}(\tau), u). \tag{52}$$

Thus, using the lemma, and taking the expression $f(x(\tau), u) + \dfrac{x_0}{t_1 - t_0}$ out from the bracket on the right, we obtain

$$\Pi(\tau, u) = \psi(\tau)\left((f(x(\tau), u) + \frac{x_0}{t_1 - t_0}\right), \tag{53}$$

where

$$\psi(\tau) = a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{ax} \times$$

$$\times \int_{\tau}^{t_1} \frac{\partial f(x(t), u(t))}{\partial x}\left(1 - \int_{\tau}^{t_1} M(t, \theta)\,d\theta\right)dt + a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}. \tag{54}$$

It only remains to show that the matrix function $\psi(\tau)$ satisfies almost everywhere on the interval $[t_0, t_1]$ the differential equation

$$\dot{\psi} = -\psi\frac{\partial f}{\partial x}. \tag{55}$$

We calculate the derivative of the function $\psi(\tau)$, using (54):

$$\dot{\psi}(\tau) = a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}\int_{\tau}^{t_1}\frac{\partial f(x(t), u(t))}{\partial x}M(t, \tau)\,dt -$$

$$- a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}\frac{\partial f(x(\tau), u(\tau))}{\partial x}. \tag{56}$$

According to the lemma, the matrix function $M(t, \tau)$ for $t_1 \geq t \geq \tau \geq t_0$ is determined from the condition

$$M(t, \tau) + \frac{\partial f(x(\tau), u(\tau))}{\partial x} = \int_{t}^{t_1} M(t, \theta)\,d\theta\,\frac{\partial f(x(\tau), u(\tau))}{\partial x}. \tag{57}$$

Transferring the term $\frac{\partial f(x(\tau), u(\tau))}{\partial x}$ from the left to the right side of the equation, and then taking it out as a common factor, we obtain

$$M(t, \tau) = \left(\int_{\tau}^{t_1} M(t, \theta)\,d\theta - 1\right)\frac{\partial f(x(\tau), u(\tau))}{\partial x}. \tag{58}$$

When we substitute this expression in (56), take the term $-\frac{\partial f(x(\tau), u(\tau))}{\partial x}$ out from under the sign of integration, and use brackets, we find that almost everywhere on $[t_0, t_1]$ we have

$$\dot{\psi}(\tau) = -\left[a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}\int_{\tau}^{t_1}\frac{\partial f(x(t), u(t))}{\partial x}\left(1 - \int_{\tau}^{t_1} M(t, \theta)\,d\theta\right)dt + \right.$$

$$\left. + a\frac{\partial\varphi\left(\int_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}\right]\frac{\partial f(x(\tau), u(\tau))}{\partial x}. \tag{59}$$

But from (54) we see that the expression in the square brackets is none other than $\psi(\tau)$. Thus (55) has been proved.

It is evident that the maximum of the function H with respect to u, considered in [1], coincides with the maximum of the function $\Pi(\tau, u)$ in (53), and the function $\psi(\tau)$ in this expression satisfies the differential equation (55). Thus the maximum principle for the problem stated at the beginning of this article has been proved.

Moreover, if in the expression (54) for the function $\psi = \psi(\tau)$, we set $\tau = t_1$, then we obtain

$$\psi(t_1) = a\frac{\partial\varphi\left(\int\limits_{t_0}^{t_1} f(x(t), u(t))\,dt\right)}{\partial x}. \tag{60}$$

This means that the vector $\psi(t_1)$ is orthogonal, at the point $x(t_1)$, to the manifold M defined by the equations (43).

We have thus proved the transversality condition obtained in [1].

As an example of optimal control of an object with distributed parameters, we consider the problem of heating a "thick" body with zero initial temperature, through a thickness $x (0 \leq x \leq 1)$ for a given time T. The condition must be satisfied that the deviation of the temperature distribution $Q = Q(x, t)$ $(0 \leq t \leq T)$ of the body from a certain given distribution $Q^*(x)$ at time T be minimal, in the sense that the functional

$$I^0 = \int\limits_0^1 [Q^*(x) - Q(x, T)]^\gamma\,dx \tag{61}$$

is a minimum. Here γ is a number greater than zero.

In this case, the controlling parameter $u = u(t)$ $(0 \leq t \leq T)$ is the temperature of the external medium which is heating the surface of the body.

It is natural that the temperature of the heating medium $u(t)$ be a bounded function, i.e., $A_1 \leq u(t) \leq A_2$ $(0 \leq t \leq T)$, where A_1, A_2 are given, that is, the set Ω in this case is the interval $[A_1, A_2]$.

As is known, the temperature distribution $Q = Q(x, t)$ $(0 \leq x \leq 1, 0 \leq t \leq T)$ can be given by the relation

$$Q(x, t) = \int\limits_0^T K(x, t, \tau)\,u(\tau)\,dt, \tag{62}$$

where $K(x, t, \tau)$ is a known scalar function.*

Since in this case, the kernel $K(x, t, \tau)$ is independent of the controlled function $Q(x, t)$, then, according to the equation (22), the function $M(t, \tau) \equiv 0$ for any t and τ in $[0, T]$. According to the theorem (p. 1166), the optimal control $u = u(\tau)$ $(0 \leq \tau \leq T)$ at any fixed instant of time τ must yield the maximum of the function

$$\Pi(\tau, u) = c_0\int\limits_0^1 \frac{\partial F^0(x, Q(x, T))}{\partial Q} K(x, T, \tau)\,u\,dx \tag{63}$$

or, in the notation of the last problem,

$$\Pi(\tau, u) = c_0\int\limits_0^1 \frac{\partial [Q^*(x) - Q]^\gamma}{\partial Q} K(x, T, \tau)\,u\,dx =$$

$$= -\gamma c_0 u\int\limits_0^1 \left[Q^*(x) - Q(x, T)\right]^{\gamma-1} K(x, T, \tau)\,dx. \tag{64}$$

* By the Green's function method.

Here $\underline{u}$ does not depend on $\underline{x}$, and can be taken out from under the integration sign.

Since $c_0 = -1$, according to the theorem, then $-\gamma c_0 > 0$, and the maximum of the function $\Pi(\tau, u)$ relative to $\underline{u}$ is evidently attained for the condition $A_1 \le u \le A_2$ when

$$u(\tau) = \frac{A_1 + A_2}{2} + \frac{A_2 - A_1}{2} \operatorname{sign} \int_0^1 \left[Q^*(x) - Q(x, T) \right]^{\gamma - 1} K(x, T, \tau)\, dx. \tag{65}$$

If, instead of the function $Q(x, T)$ in the equation (65), we substitute the expression given for it in (62), then we obtain an integral equation for the optimal control $u = u(\tau)$. For example, if $\gamma = 2$, $A_1 = -1$, $A_2 = 1$ (minimizing the square of the deviation), then

$$u(\tau) = \operatorname{sign} \int_0^1 \left[Q^*(x) - \int_0^T K(x, T, \tau)\, u(\tau)\, d\tau \right] K(x, T, \tau)\, dx. \tag{66}$$

Rearranging and changing the order of integration, we obtain

$$u(\tau) = \operatorname{sign} \left[B(\tau) - \int_0^T L(\tau, \theta)\, u(\theta)\, d\theta \right], \tag{67}$$

where $L(\tau, \theta)$ is a symmetric kernel

$$L(\tau, \theta) = \int_0^1 K(x, T, \tau)\, K(x, T, \theta)\, dx, \tag{68}$$

$$B(\tau) = \int_0^1 Q^*(x)\, K(x, T, \tau)\, dx. \tag{69}$$

This result coincides exactly with the result of solving the problem considered in [4], but it has been obtained more simply and rapidly.

We now show how the result we have obtained can be applied in the solution of a known minimax problem, which can be formulated in the following way. Let the controlled coordinate $x = x(t)$ be related to the controlling action $u = u(t)$ by the integral relation

$$x(t) = \varphi(t) + \int_0^T K(t, \tau)\, u(\tau)\, d\tau, \tag{70}$$

where we will assume that

$$\begin{aligned} &\varphi(t) \not\equiv 0 \quad \text{for} \quad 0 \leqslant t \leqslant T, \\ &K(t, \tau) \not\equiv 0 \quad \text{for} \quad 0 \leqslant \tau \leqslant t \leqslant T, \\ &K(t, \tau) \equiv 0 \quad \text{for} \quad 0 \leqslant t < \tau \leqslant T. \end{aligned} \tag{71}$$

We also assume that

$$|(u(t)| \leqslant 1 \quad \text{for} \quad 0 \leqslant t \leqslant T. \tag{72}$$

We are to find a control $u(t)$ $(0 \le t \le T)$, satisfying the condition (72), such that the functional

$$I^0 = I^0(x, (t) = \max_{0 < t < T} |x(t) \tag{73}$$

has its minimum value, whereupon

$$x(T) = 0. \qquad (74)$$

This last condition is equivalent to the condition

$$I^1 = I^1(x(t)) = \int_0^T \delta(T-\tau)\, x(\tau)\, d\tau = 0. \qquad (75)$$

The gradient of the functional I^0 is

$$\operatorname{grad} I^0(x(t)) = \delta(t_m - t), \qquad (76)$$

where t_m is the time when the function $|x(t)|$ $(0 \le t \le T)$ attains its maximum.

The gradient of the functional I^1 is

$$\operatorname{grad} I^1(x(t)) = \delta(T-\tau). \qquad (77)$$

Thus the function $\Pi(\tau, u)$ takes the form

$$\Pi(\tau, u) = c_0 \int_0^T \delta(t_m - t)\left[\frac{1}{T}\varphi(t) + K(t,\tau)u\right]dt + $$

$$+ c_1 \int_0^T \delta(T-t)\left[\frac{1}{T}\varphi(t) + K(t,\tau)u\right]dt = c_0 \frac{1}{T}\varphi(t_m) +$$

$$+ c_0 K(t_m, \tau)u + c_1 \frac{1}{T}\varphi(T) + c_1 K(T, \tau). \qquad (78)$$

Since $c_0 = -1$, the maximum of the function $\Pi(\tau, u)$ is reached for

$$u = u(\tau) = \operatorname{sign}\left[K(t_m, \tau) - c_1 K(T, \tau)\right] \quad (0 < \tau < T). \qquad (79)$$

Since for $T \ge \tau > t_m$ we have

$$K(t_m, \tau) = 0.$$

then

$$u = u(\tau) = -\operatorname{sign} c_1 K(T, \tau) \ \text{for} \ T \geqslant \tau > t_m. \qquad (80)$$

But the last formula for the control $u(\tau)$ coincides with the formula for the control in the case when the time of the transition process is minimized. Thus, after the instant $t = t_m$ when the function $|x(t)|$ $(0 \le t \le T)$ has attained its maximum value, the optimal control relative to rapidity of action does not lead to any worse value of functional I^0 than the value $|x(t_m)|$ in the optimal process minimizing this functional. We should remark that the time t_m can be determined by the method of successive approximations.

APPENDIX I

Proof of the theorem. We will assume that there exist an optimal control $u = u(S)$ $(S \in D,\ u \in \Omega)$ and an optimal state $Q = Q(S)$, such that, in view of the equation (10), the conditions (15) are satisfied:

$$I^i = 0 \quad (i = 0, 1, \ldots, p-1, p+1, \ldots, q;\ 0 < p < q)$$

and the functional I^p attains its minimum value.

We introduce the following functional:

$$I_* = \sum_{i=0}^{q} \lambda_i I^i = \lambda I = \lambda' I_1 + \lambda'' I_2, \tag{81}$$

where

$$\lambda = \| \lambda_0, \ldots, \lambda_q \|, \quad \lambda' = \| \lambda_0, \ldots, \lambda_l \|, \quad \lambda'' = \| \lambda_{l+1}, \ldots, \lambda_q \|, \tag{82}$$

$$I = \begin{Vmatrix} I^0 \\ \cdot \\ \cdot \\ \cdot \\ I^q \end{Vmatrix}, \quad I_1 = \begin{Vmatrix} I^0 \\ \cdot \\ \cdot \\ \cdot \\ I^l \end{Vmatrix}, \quad I_2 = \Phi = \begin{Vmatrix} \Phi^{l+1}(z) \\ \cdot \\ \cdot \\ \cdot \\ \Phi^q(z) \end{Vmatrix}. \tag{83}$$

Further, we set up a correspondence between each point $P \in D$ and points $S_\alpha \in N_\alpha$ ($\alpha = 1, 2, \ldots, m$), such that when the point P traverses the whole region D, then the point S_α traverses the whole set N_α ($\alpha = 1, 2, \ldots, m$).

We fix an arbitrary point $P' \in D$ and the points $S'_\alpha \in N_\alpha$ corresponding to it, which are regular for the controls $u_{\alpha\beta}(S_\alpha)$, and surround the point $S''_\alpha \in N_\alpha$ ($\alpha = 1, 2, \ldots, m$) with a region $\delta_\alpha \subset N_\alpha$, the volume of which tends to zero when the diameter d_α of the region tends to zero.

We now determine the control $\tilde{u} = \tilde{u}(S)$, giving the modification of the optimal control $u = u(S)$:

$$\tilde{u} = \tilde{u}(S) = \begin{cases} u(S) & \text{for } S_\alpha \in N_\alpha - \delta_\alpha \\ v & \text{for } S_\alpha \in \delta_\alpha \end{cases} \quad (\alpha = 1, 2, \ldots, m), \tag{84}$$

where $\underline{v}$ is an arbitrary point of the control region Ω.

The value of the functional I_* is now calculated from the modified control (84), with an accuracy up to small quantities of a higher order than ε, where $\varepsilon > 0$ tends to zero when the diameter d_α of the region $\delta_\alpha \subset N_\alpha$ ($\alpha = 1, 2, \ldots, m$) tends to zero:

$$\begin{aligned} I_*(\tilde{u}(S)) &= \lambda' I_1(Q(S) + \Delta Q(S)) + \\ &+ \lambda''\Phi\Big(\int_D F(S, Q(S) + \Delta Q(S), \tilde{u}(S))\, dS = \\ &= \lambda' I_1(Q(S) + \Delta Q(S)) + \lambda''\Phi\Big(\int_D F(S, Q(S), u(S))\, dS + \\ &+ \int_D \frac{\partial F(S, Q(S), u(S))}{\partial Q} \Delta Q(S)\, dS + \varepsilon(F(S', Q(S'), v) - \\ &- F(S', Q(S'), u(S')))). \end{aligned} \tag{85}$$

Here the increment $\Delta Q(S)$, obtained from the modification of the optimal control $u(S)$ in the equation (10), satisfies, with an accuracy up to small quantities of higher order than ε, the nonhomogeneous Fredholm integral equation, linear in $\Delta Q(S)$:

$$\begin{aligned} \Delta Q(P) &= \varepsilon(K(P, S', Q(S'), v) - K(P, S', Q(S'), u(S'))) + \\ &+ \int_D \frac{\partial K(P, S, Q(S), u(S))}{\partial Q} \Delta Q(S)\, dS. \end{aligned} \tag{86}$$

As is known [5], the solution of this equation for $\Delta Q(P)$ is given by

$$\begin{aligned} \Delta Q(P) = \varepsilon\Big[&K(P, S', Q(S'), v) - K(P, S', Q(S'), u(S')) - \\ &- \int_D M(P, S)(K(S, S', Q(S'), v) - K(S, S', Q(S'), u(S'))\, dS\Big], \end{aligned} \tag{87}$$

where the matrix function M(P, R) satisfies the equation (22)

Substituting the value of $\Delta Q(P)$ from (87) into expression (85), and expanding the functions Φ and I_1 in powers of ε, we obtain

$$I_*(\widetilde{u}(S)) = \lambda' I_1(Q(S)) + \lambda''\Phi\Big(\int_D F(S,\ Q(S),\ u(S))dS\Big) +$$
$$+\ \varepsilon\lambda'\,(\text{grad } I_1(Q(P)),\ K(P,\ S',\ Q(S'),\ v)) -$$
$$-\ K(S,\ S',\ Q(S'),\ u(S')) - \int_D M(P,\ S)\,(K(S,\ S',\ Q(S'),\ v) -$$
$$-\ K(S,\ S',\ Q(S'),\ u(S')dS) +$$
$$+\ \varepsilon\lambda''\,\frac{\partial\Phi\int_D F(S, Q(S), u(S))\,dS}{\partial z}\Big\{\int_D \frac{\partial F(P, Q(P), u(P))}{\partial Q}\times$$
$$\times\ [K(P,\ S',\ Q(S'),\ v) - K(P,\ S',\ Q(S'),\ u(S')) -$$
$$-\int_D M(P,\ S)\,(K(S,\ S',\ Q(S'),\ v) - K(S,\ S',\ Q(S'), u(S')))\,dS]\,dS +$$
$$+\ F(S',\ Q(S'),\ v) - F(S',\ Q(S'),\ u(S'))\Big\}. \tag{88}$$

Since the functional I_* takes the least possible value, then the main part of the increase in this functional for a variation in the control $u = u(S)$ will always be non-negative, i.e.,

$$\Delta I_* = I_*(\widetilde{u}(S)) - I_*(u(S)) \geqslant 0. \tag{89}$$

When we replace $I_*(\widetilde{u}(S))$ in (89) by the expression in (88) and use (24) and (81), we find that the main part of the increase in the functional can be expressed as

$$\Delta I_* = \varepsilon\,[\Pi(S',\ v) - \Pi(S',\ u(S'))] \geqslant 0 \tag{90}$$

where we have set $a = \lambda''$, $b = \lambda'$ and $\lambda_p = 1$.

Multiplying both sides of the inequality by -1, and assuming that $\varepsilon > 0$, we obtain

$$\Pi(S',\ v) \leqslant \Pi(S',\ u(S')). \tag{91}$$

Since the inequality (91) is valid for any point $v \in \Omega$, then the function $\Pi(S', u)$, defined by (24), attains a maximum with respect to the argument $\underline{u}$ for fixed S', i.e., for almost all $S' \in D$

$$\Pi(S,\ u) = H(S), \tag{92}$$

which was to be proved.

APPENDIX II

Proof of the lemma. Let

$$\mu(t,\ \tau) + \nu(t,\ \tau) = \int_{t_0}^{t} \mu(t,\ \theta)\,\nu(\theta,\ \tau)\,d\theta$$

and

$$\nu = \nu(t,\ \tau) = \begin{cases} \bar{\nu}(\tau) & \text{for } t_1 \geqslant t \geqslant \tau \geqslant t_0, \\ 0 & \text{for } t_1 \geqslant \tau > t \geqslant t_0, \end{cases}$$

where $\bar{\nu}(\tau)$ is a square-integrable function on the interval $[t_0, t_1]$.

We consider the case when $t_1 \geq t \geq \tau \geq t_0$. Then the function $\mu(t, \tau)$ must obviously satisfy the equation

$$\mu(t, \tau) + \bar{v}(\tau) = \int_{\tau}^{t_1} \mu(t, \theta)\, d\theta\, \bar{v}(\tau). \tag{93}$$

We now consider the case when $t_1 \geq \tau > t \geq t_0$. In this case, we obtain

$$\mu(t, \tau) = \int_{\tau}^{t_1} \bar{v}(\tau)\mu(t, \theta)\, d\theta. \tag{94}$$

For every fixed value of t, this is a Volterra integral equation. Because of the assumption in the lemma concerning the integrability of the square of $\bar{v}(\tau)$ in $[t_0, t_1]$, and the uniqueness of the solution of (94) [5], we have

$$\mu(t, \tau) \equiv 0 \quad \text{for} \quad t_1 \geqslant \tau > t \geqslant t_0.$$

LITERATURE CITED

1. V. G. Boltyanskii, R. V. Gamkrelidze, and L. S. Pontryagin, "The theory of optimal processes," Izv. AN SSSR, ser. matem. 24, No. 1 (1960).
2. L. I. Rozonoér, "The maximum principle of L. S. Pontryagin in the theory of optimal processes. I, II, III," Avtomatika i Telemekhanika, 20, Nos. 10, 11, 12 (1959).
3. A. G. Butkovskii and A. Ya. Lerner, "Optimal control of systems with distributed parameters," Avtomatika i Telemekhanika, 21, No. 6 (1960).
4. A. G. Butkovskii, "Optimal processes in systems with distributed parameters," Avtomatika i Telemekhanika, 22, No. 1 (1961).
5. F. Tricomi, Integral Equations [Russian translation] (Foreign Literature Press, Moscow, 1960).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

A PROBLEM IN THE SYNTHESIS OF OPTICAL SYSTEMS USING MAXIMUM PRINCIPLE

Chang Jen-wei

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1302-1308, October, 1961

Original article submitted February 28, 1961

The maximum principle is applied in the investigation of a problem in the synthesis of optimal control. The optimization criterion is given by the integral of the sum of the squares of the deviations.

1. The Problem

We consider optimal transient processes in a system of automatic control, described by the differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + b_i u \quad (i = 1, \ldots, n), \tag{1}$$

where the x_i are the phase coordinates of the system and the a_{ij} are constant coefficients. The function u describes the controlling action. Controls u are permissible that belong to the set U of piecewise-continuous functions satisfying the condition

$$|u(t)| \leqslant 1. \tag{2}$$

There is, at the present time, a powerful mathematical method–the maximum principle of L. S. Pontryagin [1]–by means of which we can synthesize optimal controls, in particular optimal controls relative to rapidity of action. As is known, the problem of synthesizing controls for optimal rapidity of action can be solved, generally speaking, only by the use of high-speed computers. Even in cases where it is possible to find the analytical form of the function $u(x_1, \ldots, x_n)$, the form of such functions is very complicated. It is therefore interesting to consider the problem of synthesizing systems that are rather simple, but at the same time optimal in a definite sense for a transient process. We consider the following problem in synthesizing an optimal control. For the system (1) it is desired to find, among all the permissible controls, an optimal control function $u(x_1, \ldots, x_n)$, such that the corresponding trajectory $[x_1(t), \ldots, x_n(t)]$ of system (1), starting at any initial position $[x_1(0), \ldots, x_n(0)]$, tends for $t \to \infty$ to the origin, and such that the functional

$$I = \int_0^\infty \left(\sum_{i=1}^{n} a_i x_i^2 + cu^2 \right) dt \tag{3}$$

converges and takes its smallest possible value.

The functional (3) – the integral of the sum of the squares of the deviations – is an indirect criterion for optimizing the quality of the transitional process. The coefficients a_i and c are weighting constants.

This type of problem in optimal control synthesis was investigated by A. M. Letov in [2], using the classical calculus of variations method. The difference is in the problem. Here the function u, describing the control action, is permitted to have discontinuities of the first kind, the presence of which will not permit the use of the classical calculus of variations method. The problem of minimizing the functional (3) for the system (1) was also investigated by A. M. Letov in [2] using the method of dynamic programming. In the present article, the synthesis problem will be solved by the maximum principle.

2. Solution of the Problem

It turns out that the problem posed above can easily be solved by the maximum principle. We first of all determine the optimal control in the form of a function of the time. The method of solution, using the maximum principle [1], is as follows.

We introduce the notation

$$f_0(x_1, \ldots, x_n, u) = \sum_{j=1}^{n} a_j x_j^2 + cu^2,$$

$$f_i(x_1, \ldots, x_n, u) = \sum_{j=1}^{n} a_{ij} x_j + b_i u \quad (i = 1, \ldots, n).$$

In addition to the phase coordinates $x_1, \ldots, x_n$ for the system (1), we introduce one extra coordinate x_0, satisfying the equation

$$\frac{dx_0}{dt} = f_0(x_1, \ldots, x_n, u).$$

In the (n + 1)-dimensional phase space X, there is a vector $\overline{x} = \{x_0, x_1, \ldots, x_n\}$ with rate of change given by

$$\frac{d\overline{x}}{dt} = f(x_1, \ldots, x_n, u), \tag{4}$$

where $f(x_1, \ldots, x_n, u)$ is a vector with coordinates $f_0(x_1, \ldots, x_n, u), \ldots, f_n(x_1, \ldots, x_n, u)$.

We now form the system of differential equations adjoint to the system (4), with the new variables $\psi = \{\psi_0, \psi_1, \ldots, \psi_n\}$:

$$\frac{d\psi_0}{dt} = -\sum_{\alpha=0}^{n} \frac{\partial f_\alpha}{\partial x_0} \psi_\alpha = 0,$$

$$\frac{d\psi_i}{dt} = -\sum_{\alpha=0}^{n} \frac{\partial f_\alpha}{\partial x_i} \psi_\alpha = -2a_i \psi_0 x_i - \sum_{j=1}^{n} a_{ji} \psi_j \quad (i = 1, \ldots, n). \tag{5}$$

We also form the Hamiltonian, which is the scalar product of the vector ψ and the velocity vector $d\overline{x}/dt$. We denote the Hamiltonian by

$$H(x, \psi, u) = \sum_{\alpha=0}^{n} \psi_\alpha \frac{dx_\alpha}{dt} = \sum_{\alpha=0}^{n} \psi_\alpha f_\alpha(x_1, \ldots, x_n, u) =$$

$$= \psi_0 \Big(\sum_{i=1}^{n} a_i x_i^2 + cu^2 \Big) + \sum_{i=1}^{n} \psi_i \Big(\sum_{j=1}^{n} a_{ij} x_j + b_i u \Big). \tag{6}$$

For fixed $x_1, \ldots, x_n$ and $\psi_0, \psi_1, \ldots, \psi_n$, the function H becomes a function of $\underline{u}$; we denote the upper bound of the values of this function by

$$M = (x, \psi) = \sup_{u \in U} H(x, \psi, u).$$

According to the theorem giving the maximum principle [1], the condition necessary for the optimization of the control u(t) is as follows. Let u(t) be the optimal control and let $x(t) = \{x_0(t), x_1(t), \ldots, x_n(t)\}$ be the corresponding trajectory of system (1). Then there exists a non-zero, continuous, vector function $\psi(t) = \{\psi_0(t), \psi_1(t), \ldots, \psi_n(t)\}$, such that the functions x(t), ψ(t), and u(t) satisfy the systems (4) and (5), while at any instant of time the Hamiltonian H is maximum relative to $\underline{u}$, i.e.,

$$H\,[x\,(t), \psi\,(t), u\,(t)] = M\,[x, (t), \psi\,(t)]. \tag{7}$$

If x(t), ψ(t), and u(t) satisfy these conditions, then

$$\psi_0\,(t) = \text{const} \leqslant 0, \quad M\,[x\,(t), \psi\,(t)] = 0. \tag{8}$$

It is easy to show, from (7) and (8), that the optimal control u(t) is given by

$$u\,(t) = \begin{cases} \frac{1}{2c}\sum_{i=1}^{n} b_i\psi_i & \text{for } \left|\frac{1}{2c}\sum_{i=1}^{n} b_i\psi_i\right| \leqslant 1, \\ 1 & \text{for } \frac{1}{2c}\sum_{i=1}^{n} b_i\psi_i \geqslant 1, \\ -1 & \text{for } \frac{1}{2c}\sum_{i=1}^{n} b_i\psi_i \leqslant -1. \end{cases} \tag{9}$$

We note that in the right-hand side of (6), the terms containing $\underline{u}$ in the expression for H are

$$c\psi_0\,(t)u^2\,(t) + u\,(t)\sum_{i=1}^{n} b_i\psi_i\,(t). \tag{10}$$

The expression (10) evidently has a maximum at the same time as the function H for fixed $\psi_0, \ldots, \psi_n$; $x_1, \ldots, x_n$. Therefore the necessary condition (7) becomes

$$c\psi_0\,(t)\,u^2\,(t) + u\,(t)\sum_{i=1}^{n} b_i\psi_i\,(t) = \sup_{u\in U}\left[c\psi_0\,(t)\,u^2 + u\sum_{i=1}^{n} b_i\psi_i\,(t)\right]. \tag{11}$$

We rewrite the expression (11) in the form

$$c\psi_0\,(t)u^2 + u\sum_{i=1}^{n} b_i\psi_i\,(t) = c\psi_0\,(t)\left[u + \frac{1}{2c\psi_0}\sum_{i=1}^{n} b_i\psi_i\,(t)\right]^2 - \frac{1}{4c\psi_0}\left[\sum_{i=1}^{n} b_i\psi_i\,(t)\right]^2.$$

The function H (x, ψ, u) in (6) is homogeneous relative to ψ_α (α = 0, 1, . . . , n). Therefore, according to the condition (8), we may assume that ψ_0 (t) = −1. It is then easy to see that if u(t) is determined according to (9), the expression (11) attains its maximum.

Now, having the expression for u(t) in (9), we can establish the relation between the functions $\psi_1(t), \ldots, \psi_n(t)$ and the functions $x_1(t), \ldots, x_n(t)$, and this directly determines the optimal control $\underline{u}$ in terms of the coordinates $x_1, \ldots, x_n$. We first of all assume that after the time of the transition process, the control $\underline{u}$ has not reached its limiting value. Then, substituting the first expression in (9) in the right-hand sides of the system of equations (5), we obtain the linear system of 2n differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}x_j + \frac{b_i}{2c}\sum_{j=1}^{n} b_j\psi_j, \quad \frac{d\psi_i}{dt} = 2a_i x_i - \sum_{j=1}^{n} a_{ji}\psi_j \quad (i = 1, \ldots, n) \tag{12}$$

in the 2n unknown functions $x_1(t), \ldots, x_n(t), \psi_1(t), \ldots, \psi_n(t)$. The system (12) can be solved for given values of the boundary conditions. For given, fixed initial conditions $x_1(0), \ldots, x_n(0)$, we must choose the corresponding initial values $\psi_1(0), \ldots, \psi_n(0)$, so that the solution $x_1(t), \ldots, x_n(t)$ of the system (12) tends to zero for $t \to \infty$. The characteristic equation of the system (12) is

$$\Delta(\lambda) = \begin{vmatrix} a_{11} - \lambda \ldots a_{1n} & \frac{b_1^2}{2c} \ldots \frac{b_1 b_n}{2c} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ a_{n1} \ldots a_{nn} - \lambda & \frac{b_1 b_n}{2c} \ldots \frac{b_n^2}{2} \\ 2a_1, \ldots 0 & -a_{11} - \lambda \ldots - a_{n1} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ 0 \ldots 2a_n & -a_{1n} \ldots - a_{nn} - \lambda \end{vmatrix} = 0. \tag{13}$$

Using elementary methods of transformation of the determinant (13), we can show that the relation $\Delta(\lambda) = \Delta(-\lambda) = 0$ holds, i.e., the equation has roots in pairs, equal in magnitude but with opposite signs. We will assume that these roots are simple, and will denote them by $\lambda_1, \ldots, \lambda_n; -\lambda_1, \ldots, -\lambda_n$. Let the condition $\mathrm{Re}\,\lambda_i < 0$ $(i = 1, \ldots, n)$ be satisfied, which is evidently always possible. Then the general solution of the system (12) can be written in the form

$$x_i(t) = \sum_{j=1}^{n} c_j \Delta_i(\lambda_j) e^{\lambda_j t} + \sum_{j=1}^{n} c_{n+j} \Delta_i(-\lambda_j) e^{-\lambda_j t}, \tag{14}$$

$$\psi_i(t) = \sum_{j=1}^{n} c_j \Delta_{n+i}(\lambda_j) e^{\lambda_j t} + \sum_{j=1}^{n} c_{n+j} \Delta_{n+i}(-\lambda_j) e^{-\lambda_j t}, \tag{15}$$

$$(i = 1, \ldots, n)$$

where the $\Delta_k(\lambda_j)$ $(k = 1, 2, \ldots, 2n;\ j = 1, \ldots, n)$ are $(n-1)$th order minors of the rows of the determinant $\Delta(\lambda_j)$ for which they are not all zero [3].

The arbitrary constants $c_1, \ldots, c_{2n}$ are obtained from the initial values $x_1(0), \ldots, x_n(0); \psi_1(0), \ldots, \psi_n(0)$. In order that for $t \to \infty$ the functions $x_1(t), \ldots, x_n(t)$ tend to zero, it is necessary that $c_{n+j} = 0$ $(j = 1, \ldots, n)$. It can be seen from (15) that the c_{n+j} are zero if

$$\psi_i(0) = \sum_{j=1}^{n} c_j \Delta_{n+i}(\lambda_j) \quad (i = 1, \ldots, n) \tag{16}$$

(we assume that $|\Delta_{n+i}(-\lambda_j)| \neq 0$).

On the other hand, the arbitrary constants c_j $(j = 1, \ldots, n)$ must satisfy the conditions

$$x_i(0) = \sum_{j=1}^{n} c_j \Delta_i(\lambda_j) \quad (i = 1, \ldots, n). \tag{17}$$

When we eliminate c_j from (16) and (17), we obtain the relation between $x_i(0)$ and $\psi_i(0)$:

$$\psi_i(0) = \sum_{j=1}^{n} \gamma_{ij} x_j(0) \quad (i = 1, \ldots, n). \tag{18}$$

It should be remarked that this relation must hold at some instant of time, and we can always take this instant for the initial time. Thus the optimal control $\underline{u}$ inside the boundary has been obtained in the form

$$u = \frac{1}{2c} \sum_{i=1}^{n} k_i x_i, \tag{19}$$

where

$$k_i = \sum_{j=1}^{u} b_j \gamma_{ji}.$$

It is obvious that the relation (19) only holds in the case when the absolute value of the sum $\frac{1}{2c}(k_1x_1+\ldots$ $\ldots+k_nx_n)$ is not greater than one. We can easily see that when $\left|\frac{1}{2c}(k_1x_1+\ldots+k_nx_n)\right|>1$, then the optimal control must be beyond the boundary(i.e., $|u| = 1$)if the right-hand boundary conditions are fulfilled, that is, the solution of system (1), which is nonlinear due to the nonlinear relation between $\underline{u}$ and $x_1,\ldots,x_n$, tends to zero. In other words, solution of this system must be asymptotically stable. Thus the optimal control is given by

$$u=\begin{cases}\frac{1}{2c}\sum_{i=1}^{n}k_ix_i & \text{for } \left|\frac{1}{2c}\sum_{i=1}^{n}k_ix_i\right|\leqslant 1,\\ 1 & \text{for } \frac{1}{2c}\sum_{i=1}^{n}k_ix_i\geqslant 1,\\ -1 & \text{for } \frac{1}{2c}\sum_{i=1}^{n}k_ix_i\leqslant -1.\end{cases}\tag{20}$$

The question of the stability of the system (1) with the control (20) has to be investigated separately. We see that, because of the continuity, stability is guaranteed in the region $\left|\frac{1}{2c}(k_1x_1+\ldots+k_nx_n)\right| = 1+\varepsilon$, where ε is sufficiently small. The question of the stability of the system (1) with the control (20) thus reduces to the problem of finding a region of attraction, the boundaries of which lie beyond the limits $\left|\frac{1}{2c}(k_1x_1+\ldots+k_nx_n)\right| = 1$. This problem will be the object of a separate investigation.

3. Comments

It is interesting to compare the result we have obtained with the result found in [2] by the classical method of calculus of variations, for a more restricted class of control functions than we have used in the present article. We find that the results are the same. The system (12) coincides with the Euler system of equations (3.3) in [2]; in the former, the functions $\psi_i(t)$ play the part of Lagrange multipliers, and the results are obtained by solving these systems of differential equations. Thus no new results are obtained for the problem by generalizing the class of permissible functions.

The author wishes to thank A. M. Letov for several valuable suggestions which were of use in the present article.

4. Example

We consider the system of the second order

$$\frac{d^2x}{dt^2}+2a\frac{dx}{dt}+\omega^2x=u \quad (|u|\leqslant 1).\tag{21}$$

The function to be minimized is

$$\int_0^\infty (ax^2+cu^2)dt.$$

When we transform (21) to the normal form, we obtain the system in the form

$$\begin{aligned}\frac{dx_1}{dt}&=x_2, & \frac{d\psi_1}{dt}&=2ax_1+\omega^2\psi_2,\\ \frac{dx_2}{dt}&=\omega^2x_1-2ax_2+\frac{1}{2c}\psi_2, & \frac{d\psi_2}{dt}&=-\psi_1+2a\psi_2.\end{aligned}\tag{22}$$

The roots of the characteristic equation for the system (22) are

$$\lambda_1=-\alpha+i\beta,\ \lambda_2=-\alpha-i\beta,\ \lambda_3=\alpha+i\beta,\ \lambda_4=\alpha-i\beta,$$

where $i=\sqrt{-1}$, $\beta=\sqrt{\omega^2-\alpha}$.

The general solution of the system (22) has the form [3]

$$
\begin{aligned}
x_1 &= c_1x_{11} + c_2x_{12} + c_3x_{13} + c_4x_{14},\\
x_2 &= c_1x_{21} + c_2x_{22} + c_3x_{23} + c_4x_{24},\\
\psi_1 &= c_1\psi_{11} + c_2\psi_{12} + c_3\psi_{13} + c\psi_{14},\\
\psi_2 &= c_1\psi_{21} + c_2\psi_{22} + c_3\psi_{23} + c_4\psi_{24}.
\end{aligned} \tag{23}
$$

Here

$$
\begin{aligned}
x_{11} &= e^{-\alpha t}(\alpha\cos\beta t + \beta\sin\beta t), & x_{12} &= e^{-\alpha t}(\alpha\sin\beta t - \beta\cos\beta t),\\
x_{13} &= \frac{1}{1c}e^{\alpha t}\cos\beta t, & x_{14} &= \frac{1}{2c}e^{\alpha t}\sin\beta t,\\
x_{21} &= e^{-\alpha t}((\beta^2-\alpha^2)\cos\beta t - 2\alpha\beta\sin\beta t), & x_{22} &= e^{-\alpha t}(2\alpha\beta\cos\beta t + (\beta^2-\alpha^2)\sin\beta t),\\
x_{23} &= e^{\alpha t}\left(\frac{\alpha}{2c}\cos\beta t - \frac{\beta}{2c}\sin\beta t\right), & x_{24} &= e^{\alpha t}\left(\frac{\beta}{2c}\cos\beta t + \frac{\alpha}{2c}\sin\beta t\right),\\
\psi_{11} &= e^{-\alpha t}\left(\frac{-3a}{2}\cos\beta t - \frac{a\beta}{2\alpha}\sin\beta t\right), & \psi_{12} &= e^{-\alpha t}\left(\frac{a\beta}{2\alpha}\cos\beta t - \frac{3a}{2}\sin\beta t\right),\\
\psi_{13} &= 4a\omega^2e^{\alpha t}\cos\mu t, & \psi_{14} &= 4a\,\omega^2e^{\alpha t}\sin\beta t,\\
\psi_{21} &= \frac{-a}{2\alpha}e^{-\alpha t}\cos\beta t, & \psi_{22} &= \frac{-a}{2\alpha}e^{-\alpha t}\sin\beta t,\\
\psi_{23} &= e^{\alpha t}(4\alpha^2\cos\beta t - 4\alpha\beta\sin\beta t), & \psi_{24} &= e^{\alpha t}(4\alpha\beta\cos\beta t + 4\alpha^2\sin\beta t).
\end{aligned}
$$

In order that x_1 and x_2 tend to zero for $t \to \infty$, it is necessary to choose the initial values $\psi_1(0)$ and $\psi_2(0)$ so that

$$
\psi_1(0) = c_1\psi_{11}(0) + c_2\psi_{12}(0) = -\frac{3a}{2}c_1 + \frac{a\beta}{2\alpha}c_2, \quad \psi_2(0) = c_1\psi_{21}(0) + c_2\psi_{22}(0) = -\frac{a}{2\alpha}c_1. \tag{24}
$$

Actually, since the determinant

$$
\begin{vmatrix}\psi_{13}(0) & \psi_{14}(0)\\ \psi_{23}(0) & \psi_{24}(0)\end{vmatrix} = \begin{vmatrix}4a\omega^2 & 0\\ 4\alpha^2 & 4\alpha\beta\end{vmatrix}
$$

is not zero, it follows from the last two equations of the system (23) that $c_3 = c_4 = 0$.

In this case, the equalities

$$
\begin{aligned}
x_1(0) &= c_1x_{11}(0) + c_2x_{12}(0) = \alpha c_1 - \beta c_2,\\
x_2(0) &= c_1x_{21}(0) + c_2x_{22}(0) = (\beta^2-\alpha^2)c_1 + 2\alpha\beta c_2
\end{aligned} \tag{25}
$$

will hold.

If we eliminate c_1 and c_2 from (24) and (25), we obtain the relations between $\psi_1(0)$, $\psi_2(0)$, and $x_1(0)$, $x_2(0)$:

$$
\psi_1(0) = \frac{-a}{\omega^2}\left[(5\alpha^2+\beta^2)\frac{1}{2\alpha}x_1(0) + x_2(0)\right], \quad \psi_2(0) = \frac{-a}{\omega^2}\left(x_1(0) + \frac{1}{2\alpha}x_2(0)\right).
$$

Hence the optimal control is obtained in the form

$$
u(x_1, x_2) = \begin{cases} -\dfrac{a}{2c\omega^2}\left(x_1 + \dfrac{1}{2\alpha}x_2\right) & \text{for } \left|-\dfrac{a}{2c\omega^2}\left(x_1 + \dfrac{1}{2\alpha}x_2\right)\right| < 1,\\ 1 & \text{for } -\dfrac{a}{2c\omega^2}\left(x_1 + \dfrac{1}{2\alpha}x_2\right) \geqslant 1,\\ -1 & \text{for } -\dfrac{1}{2c\omega^2}\left(x_1 + \dfrac{1}{2\alpha}x_2\right) < -1. \end{cases}
$$

LITERATURE CITED

1. V. G. Boltyanskii, R. V. Gamkrelidze, and L. S. Pontryagin, "The theory of optimal processes.I," Izv. AN SSSR, 24, No. 1 (1960).
2. A. M. Letov, "The analytic construction of results. I, II, IV," Avtomatika i Telemekhanika, 21, Nos. 4, 5 (1960); 22, No. 4 (1961).
3. V. V. Stepanov, A Course in Differential Equations [in Russian] (Fizmatgiz, 1958).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

THE CONNECTION BETWEEN S. A. CHAPLYGIN'S THEOREM AND THE THEORY OF OPTIMAL PROCESSES

A. I. Averbukh

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1309-1313, October, 1961

Original article submitted April 11, 1961

The connection between S. A. Chaplygin's theorem on differential inequalities and the theory of optimal control is considered for the purpose of obtaining a necessary condition for the applicability of this theorem in a given interval, and also in order to obtain a statement concerning the dependence of the magnitude of this interval on the character of the variables in the problem.

If two functions y(x) and z(x), which coincide together with their first n−1 derivatives at $x = X_0$, satisfy the differential equation $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ and the inequality $z^{(n)} \geq f(x, z, z', \ldots, z^{(n-1)})$ respectively, then clearly on some interval $[X_0, X_1]$ they satisfy $z(x) \geq y(x)$. The theorem of S. A. Chaplygin on differential inequalities consists in indicating an upper limit for X_1 for linear equations. It is as follows: on the segment $[X_0, X_1]$ the solution of the adjoint equation with zero initial conditions on the derivatives and positive initial condition at $X = X_1$ on the unknown function itself should be nonnegative [1, 5, 6].

In the case of nonlinear equations the matter is significantly more complicated. B. N. Petrov [2] constructed an example of a nonlinear equation such that for any $\varepsilon > 0$ there exists a comparison function z(x) for which $X_1 < \varepsilon$, since for $X \geq X_1$ already $z(x) < y(x)$; so that the upper limit of X_1 is equal to X_0.

The problem* of finding conditions under which $z(x) \geq y(x)$ on the segment $[X_0, X_1]$, if

$$y^{(n)} = f(x, \ldots, y^{(n-1)}), \quad z^{(n)} \geqslant f(x, \ldots, z^{(n-1)}),$$

may be solved with the aid of methods used in the theory of optimal systems [3].

In fact, if, on the segment $[X_0, X_1]$ the conditions

$$\begin{aligned} &y_i' = f_i(x, y_1, \ldots, y_n), \\ &z_i' \geqslant f_i(x, z_1, \ldots, z_n), \quad (i = 1, \ldots, n) \\ &y_i(X_0) = z_i(X_0) = y_{i0} \end{aligned} \tag{1}$$

imply that

$$z_i \geqslant y_i \quad (i = 1, \ldots, n), \tag{2}$$

then for all $u_i(x) \geq 0$ the solutions of the system

$$\begin{aligned} &y_i' = f_i(x, y_1, \ldots, y_n), \\ &z_i' = f_i(x, y_1, \ldots, z_n) + u_i(x), \\ &y_i(X_0) = z_i(X_0) = y_i^0 \end{aligned} \tag{3}$$

satisfy the inequality

* In the following it will be termed Chaplygin's problem.

$$y_i(x) \leqslant z_i(x) \quad (i = 1, \ldots, n) \tag{4}$$

for all $x \in [X_0, X_1]$.

To determine the conditions under which (4) is fulfilled, we use the fact that $z_i(x)$ is a functional of $u(x) = \{u_1(x), \ldots, u_n(x)\}$, and $y_i(x)$ is the value of this functional for $u(x) \equiv 0$. Consequently, it is necessary to determine the variation in values of the functional when the control is varied from $u \equiv 0$.

In [4] a formula is obtained giving the increment of the value of the functional

$$S = \Sigma c_i x_i(t),$$

where the functions $x_i(t)$ satisfy the system

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_n, t) \quad (i = 1, \ldots, n, \ x_i(0) = x_i). \tag{5}$$

If the control $u(t)$ is changed from $v_i^0(t)$ to $v_i^{(1)}(t) = v_i^{(0)}(t) + \delta u(t)$, then

$$\delta S = \sum_{n=1}^{n} c_i \delta x_i(T) = -\int_0^T [H(x, p, v^{(1)}, (t) - H(x, p, v^{(0)}, t)]\, dt - \eta, \tag{6}$$

where

$$H(x, p, u, t) = \sum_{i=1}^{n} p_i f_i(x, u, t), \tag{7}$$

and $x_i(t)$ and $p_i(t)$ are solutions of the canonical system

$$\dot{x}_i = \frac{\partial H(x, p, v^0, t)}{\partial p_i} = f_i(x, t, v^{(0)}),$$
$$\dot{p}_i = -\frac{\partial H(x, p, v^{(0)}, t)}{\partial x_i} \quad (i = 1, \ldots, n). \tag{8}$$

The quantity $\eta(t)$ represents a residual term for which the estimate

$$|\eta| \leqslant A \int_0^{\tau} \sum_1^n \delta u_k^2 dt$$

is deduced in [4]; here A is some constant which does not depend on the quantities δu_k, and τ is a segment of the t axis on which the quantities δu_k are different from zero.

The problem (3) considered is written with the help of formulas of the following form:

$$v^{(0)}(x) \equiv 0, \quad v^{(1)}(x) \equiv u(x),$$
$$\delta x_k(x) = z_k(x) - y_k(x),$$
$$c_k = 1 \text{ and } c_j = 0 \text{ for } j \neq k.$$

In this case

$$z_k(X) - y_k(X) = -\int_{x_0}^{X} \sum_1^n p_i u_i dx - \eta, \tag{9}$$

where

$$y' = f_i(y_1, \ldots, y_n, x),$$

$$p_i' = -\sum_{s=1}^{n} p_s \frac{\partial f_s(y_1, \ldots, y_n, x)}{\partial y_i} \quad (i = 1, \ldots, n), \tag{10}$$

$$p_k(X) = -c_k, \ p_j(X) = 0 \text{ for } j \neq k \quad (k = 1, \ldots, n). \tag{11}$$

The solution of Chaplygin's problem reduces to finding conditions under which for any $u_i(x) \geq 0$ the right side of (9) is nonnegative. Carrying through exactly the reasoning in [4] which leads to the proof of the maximum principle, and considering the estimate for η, we obtain a necessary condition for the nonnegativity of $z_k(x)-y_k(x)$ for all $x \in [X_0, X_1]$ and $u_k(x) \geq 0$ as follows: the functions $p_i(x)$ which satisfy the system (10) should be nonpositive on the segment $[X_0, X_1]$ for all systems of initial conditions (11).

Thus the following is a necessary condition for the applicability of Chaplygin's theorem on the segment $[X_0, X_1]$ for the system (1): for any k and any $X \in [X_0, X_1]$ the quantities $p_{i,k}(x)$, which are solutions of n systems

$$y_i' = f_i(x, y_1, \ldots, y_n),$$

$$p_{i,k}' = -\sum_{\nu=1}^{n} p_{\nu,k} \frac{\partial f_\nu}{\partial y_i} \quad (i = 1, \ldots, n), \tag{12}$$

$$p_{i,k}(X) = -c_{i,k}\delta_{i,k} \; (c_{i,k} > 0) \quad (k = 1, \ldots, n),$$

should be nonpositive on $[X_0, X]$.

Here $\delta_{i,k}$ is the Kronecker symbol ($\delta_{i,k} = 1$ for $i = k$ and $\delta_{i,k} = 0$ for $i \neq k$).

We note that the equations for $p_{i,k}$ form a system adjoint to the system of variational equations associated with the original system (1).

From the arbitrariness of $| c_{i,k} |$ and the linearity of the system (12) it follows that for the applicability of Chaplygin's theorem on the segment $[X_0, X_1]$ it is necessary that the solutions $p_i(x)$ of the system adjoint to the variational system possess the property that for any $X \in [X_0, X_1]$, $p_i(x) < 0$ for $x \in [X_0, X]$, if $p_i(X) < 0$ for all $i = 1, \ldots, n$.

In particular for $n = 2$ it follows from this condition that of the two functions $p_1(x)$, $p_2(x)$ which are solutions of system (12), only one may change sign (for any initial conditions), and then only once.

In the case of a system linear in y_i, it is shown in [4] that $\eta \equiv 0$ and the condition mentioned is also sufficient.

If it is required that the condition $y_i \leq z_i$ be satisfied not for all i, but only for some of them, then an analogous condition is obtained, but imposed only on some of the variables. In particular for the case of a single equation of nth order it is necessary that the solution of the adjoint equation with zero initial conditions on the derivatives at $x = X$ not change sign on the segment $[X_0, X]$ for any $X \in [X_0, X_1]$. These conditions coincide with the conditions in [1, 5, 6]. In the case of a linear equation of nth order this condition is also sufficient.

In a completely analogous manner one may investigate the following generalized Chaplygin's problem ("extension of Chaplygin's theorem"): for which conditions do the inequalities

$$u_i(x) \geqslant v_i(x) \geqslant 0 \tag{13}$$

imply for any $u_i(x)$, $v_i(x)$ the inequality

$$z_i(x) \geqslant y_i(x) \tag{14}$$

for solutions of the system

$$y_i' = f_i(y_1, \ldots, y_n, x) + v_i, \; z' = f_i(z_1, \ldots, z_n, x) + u_i \; (i = 1, \ldots, n) \tag{15}$$

with initial conditions $z_i(X_0) = y_i(X_0) = y_i^0$.

A necessary condition for the satisfaction of (14) on the segment $[X_0, X_1]$ with (15) holding is the nonpositivity on $[X_0, X]$ for any $X \in [X_0, X_1]$ and any k of the quantities $p_{i,k}(x)$, solutions of the n systems of equations

$$y_i' = f_i(y_1, \ldots, y_n, x) + u_i(x),$$

$$p'_{i,k} = -\sum_{\nu=1}^{n} p_{\nu,k} \frac{\partial f_\nu}{\partial y_i} \quad (i = 1, \ldots, n;\ k = 1, \ldots, n),$$

$$p_{i,k}(X) = -c_{i,k}\,\delta_{i,k}\,(c_{i,k} > 0), \quad y_i(X_0) = y_i^0 \tag{16}$$

for any

$$u_i(x) \geqslant 0.$$

In this case the applicability of Chaplygin's theorem depends on the behavior of the collection of all solutions of the system (16) obtained for all admissible values of $u_i(x)$, whereas in the usual Chaplygin's problem only the behavior of the solutions for $u_i(x) \equiv 0$ is important. This fact is noted in [7].

With the help of the method indicated one may investigate the question of the dependence of the interval of applicability of Chaplygin's theorem on the variables of the problem. In other words, one may solve the problem of the possibility of expanding this interval by means of a suitable change of variables $y_1, \ldots, y_n$.

We introduce new variables

$$y_{n+i} = \varphi_i(y_1, \ldots, y_n, x) \quad (i = 1, \ldots, n).$$

Then instead of (12) we obtain $\underline{n}$ systems

$$y'_i = f_i(y_1, \ldots, y_n, x) + u_i,$$

$$y'_{n+i} = \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial y_\nu}(f_\nu + u_\nu) + \frac{\partial \varphi_i}{\partial x},$$

$$p'_{n+i,k} = 0 \quad (i = 1, \ldots, n;\ k = 1, \ldots, n), \tag{17}$$

$$p'_{i,k} = -\left\{\sum_{\nu=1}^{n} p_{\nu,k} \frac{\partial f_\nu}{\partial y_i} + \sum_{\nu=1}^{n} p_{n+\nu,k} \times \right.$$

$$\left. \times \left[\sum_{l=1}^{n} \frac{\partial^2 \varphi_\nu}{\partial y_l \partial y_i}(f_l + u_l) + \sum_{l=1}^{n} \frac{\partial \varphi_\nu}{\partial y_l}\frac{\partial f_k}{\partial y_i} + \frac{\partial^2 \varphi_\nu}{\partial x\, \partial y_i}\right]\right\}$$

with initial conditions

$$p_{i,k}(X) = 0, \quad p_{n+i,k}(X) = -\delta_{i,k},$$

$$y_i(X_0) = y_i^0, \quad y_{n+i}(X_0) = \varphi_i(y_1^0, \ldots, y_n^0, X_0).$$

The conditions for the applicability of Chaplygin's theorem will be the nonpositivity on $[X_0, X]$ for all $X \in [X_0, X_1]$ and all $\underline{k}$ of the quantities

$$w_{i,k} = p_{i,k} + \sum_{\nu=1}^{n} \frac{\partial \varphi_\nu}{\partial y_i} p_{n+\nu,k}, \tag{18}$$

formed from p_i, p_{n+i} and y_i and satisfying system (17). This follows from the expression similar to (9) for the system (17). For the case of Chaplygin's theorem in the usual sense it is necessary to take $u_i \equiv 0$ in (17), and for the extended theorem of Chaplygin, to consider all $u_i \geq 0$.

From (17) it follows that the quantities $w_{i,k}$ satisfy $\underline{n}$ systems of differential equations:

$$w'_{i,k} = -\sum_{\nu=1}^{n} w_{\nu,k} \frac{\partial f_\nu}{\partial y_i} \quad (i = 1, \ldots, n;\ k = 1, \ldots, n) \tag{19}$$

with initial conditions

$$w_{i,k}(X) = -\frac{\partial \varphi_k}{\partial y_i} \tag{19a}$$

(since $p_{n+i,k} \equiv -\delta_{i,k}$).

Comparing (19) with (12), one sees that the equations for $w_{i,k}$ are the same equations adjoint to the system of variational equations, as are the equations for $p_{i,k}$ in (12). From this it is clear that if the functions φ_i are such that $\partial\varphi_i / \partial y_k$ do not change sign for any $\underline{i}$ and $\underline{k}$, then it is not possible to expand the region of applicability of Chaplygin's theorem by transforming variables. The case when $\partial \varphi_i/\partial y_k$ may vanish for certain $\underline{y}$ is more complicated and is not considered here.

In conclusion the author expresses deep thanks to L. I. Rozonoér for his interest, great help, and valuable counsel as regards making the bases of this article sounder.

LITERATURE CITED

1. S. A. Chaplygin, A New Method of Approximate Integration of Differential Equations. Collection of Works [in Russian] (Gostekhizdat, 1948), Vol. 1.
2. N. N. Luzin, On Acad. S. A. Chaplygin's Method for Approximate Integration. Uspekhi Mat. Nauk, 6, No. 6 (1951).
3. L. S. Pontryagin, The Maximum Principle in the Theory of Optimal Systems. Uspekhi Mat. Nauk. 14, No. 1 (1959).
4. L. I. Rozonoér, L. S. Pontryagin's Maximum Principle and the Theory of Optimal Systems I, II, III. Avtomatika i Telemekhanika, 20, Nos.10, 11, 12 (1959).
5. N. A. Koshcheev, Exact Limits of Applicability of Chaplygin's Method for Linear Differential Equations. Dokl. Akad. Nauk SSSR, 3, No. 5 (1956).
6. N. A. Koshcheev, The Limits of Applicability of S. A. Chaplygin's Method. Uch. Zap. Kuibysh. Gos. Ped. Inst., No. 14 (1956).
7. B. N. Babkin, Approximate Integration of System of Differential Equations by Chaplygin's Method. Izv. AN SSSR, ser. met. 18, No. 5 (1954).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

POSSIBILITIES FOR SUPPRESSION OF DISTURBANCES IN A CERTAIN CLASS OF DYNAMIC SYSTEMS

M. V. Meerov

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1314-1323, October, 1961
Original article submitted March 29, 1961

The connection between the structure of a dynamic system and the intensity of disturbance effects is discussed as depending on the points of entrance of the disturbances into the system. It is shown that for a class of structures which remain stable with unlimited increase of amplification, disturbances can be suppressed with any degree of accuracy regardless of the point where they have first been applied to the system—except at the main input. A procedure for carrying out the design of structures incorporating this property is given. The problem has been solved for systems of multiarticulated controls, as well as for systems with a single controlled function.

The most important problem in the contemporary theory and practice of automatic control is choosing the design and parameters of the systems in such a way that the effects of disturbances will be at a minimum. A good deal of attention has been and continues to be given to this problem in modern literature [1-7]. In these papers, basically the case is investigated when a signal together with some disturbances is applied to the input of the system, and the problem consists in separating out the useful signal from the background of disturbances. In many automatic control systems, and in a number of tracking systems the useful signal is not accompanied by any disturbance at the input. In these cases, however, disturbances are applied at various intermediate points of the control circuit. They consist of irrelevant loads and of perturbations of various kinds, some of which may be also of a random nature.

In this paper just such a case is discussed. It is shown that for one class of structures (namely those which admit limitless increase of amplification without harm to their stability), under certain conditions specified below, suppression of disturbances consisting of irrelevant loads or of perturbations can be carried out by increase in amplification of several components of the control circuit. It is made clear how the effect of a disturbance depends on the point of its application, as well as on the structure of the entire control chain. The problem is solved for systems having a single controlled quantity or function, and also for systems having several interrelated controlled elements (the latter being sometimes called multiarticulate systems) [11, 12]. In the case considered in this paper the disturbance can be applied at any place except at the input of the signal.

1. System with a Single Quantity to be Controlled

In Fig. 1 a block diagram of an automatic control system of this kind is presented. A useful signal x_{st} (standard test signal) is applied to the system, and the latter is required to reproduce it. It is assumed that x_{st} is free of any disturbances. To simplify the following discussion, we shall assume that our system consists of four dynamic components. This does not make the conclusions which will be reached below any less valid in the general case of many components: Indeed a derivation which is valid in the general case has been given in paper [13].

The transfer function of the system component $\underline{i}$ not subjected to the effect of a disturbance is designated by $K_iR_i(p)/D_i(p)$. With respect to components upon which disturbances are acting, it is assumed for the sake of generality that the points of application of the disturbances do not coincide with the points of application of the useful signal, and the transfer functions for the disturbances are designated by $K'_iR'_i(p)/D'_i(p)$.

A condition which must be fulfilled without fail by all systems of the class under consideration is that the first component, that is, the one to which the useful signal is applied, must not be subjected to the action of any disturbances. It is assumed that the amplification factors of the components which are not directly subjected to the action of disturbances can vary within wide limits and assume sufficiently large values.

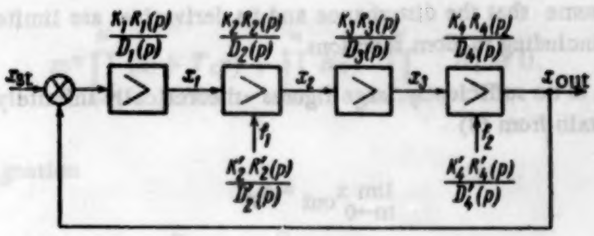


Fig. 1

Let us find the transfer function for the complete system shown in Fig. 1. We have

$$x_1 = \frac{K_1R_1(p)}{D_1(p)}[x_{st} - x_{out}], \tag{1}$$

$$x_2 = \frac{K_2R_2(p)}{D_2(p)}x_1 + \frac{K'_2R'_2(p)}{D'_2(p)}f_1, \tag{2}$$

$$x_3 = \frac{K_3R_3(p)}{D_3(p)}x_2, \tag{3}$$

$$x_{out} = \frac{K_4R_4(p)}{D_4(p)}x_3 + \frac{K'_4R'_4(p)}{D'_4(p)}f_2. \tag{4}$$

After exclusion of x_1, x_2, and x_3 from equations (1) to (4), and a few transformations, we obtain the following expression for x_{out}:

$$x_{out} = \frac{\prod_{i=1}^{4} R_i(p)\, D'_2(p)\, D'_4(p)\, K_1K_2K_3K_4x_{st}}{\prod_{i=1}^{4} D_i(p)\, D'_2(p)\, D'_4(p) + D'_2(p)\, D'_4(p) \prod_{i=1}^{4} R_i(p)\, K_1K_2K_3K_4} +$$

$$+ \frac{D_1(p)\, D_2(p)\, D'_4(p)\, R_3(p)\, R_4(p)\, R'_2(p)\, K'_2K_3K_4f_1 + \prod_{i=1}^{4} D_i(p)\, D'_2(p)\, R'_4(p)\, K'_4f_2}{\prod_{i=1}^{4} D_i(p)\, D'_2(p)\, D'_4(p) + D'_2(p)\, D'_4(p) \prod_{i=1}^{4} R_i(p)\, K_1K_2K_3K_4}. \tag{5}$$

We divide the numerator and denominator of (5) by K_1K_3. Designating $1/K_1 = 1/K_3 = m$, we then obtain the following expression for x_{out}:

$$\left[m^2 \prod_{i=1}^{4} D_i(p)\, D'_2(p)\, D'_4(p) + D'_2(p)\, D'_4(p) \prod_{i=1}^{4} R_i(p)\, K_2K_4\right] x_{out} =$$

$$= D'_2(p)\, D'_4(p) \prod_{i=1}^{4} R_i(p)\, K_2K_4x_{st} +$$

$$+ mD_1(p)\, D_2(p)\, D'_4(p)\, R_3(p) R_4(p)\, R'_2(p)\, K'_2K_4f_1 +$$

$$+ m^2 \prod_{i=1}^{4} D_i(p)\, D'_2(p)\, R'_4(p)\, K'_4f_2. \tag{6}$$

From here on we shall assume that the disturbance and its derivatives are limited in modulus; otherwise, they may be any functions of time, including random functions.

Let us consider K_1 and K_3 to be sufficiently large figures—theoretically infinitely large. If the system remains stable when $m \to 0$, then we obtain from (6)

$$\lim_{m \to 0} x_{out} = x_{st}.$$

Putting it differently, when the system remains stable with unlimited increase of amplification in the components which are not subjected to direct action of the disturbances, then it is possible to obtain as exact a reproduction of the useful signal as desired. Formula (6) shows also that disturbances are the more strongly suppressed, the farther the components to which they are applied are from the beginning of the system. The basic principle of this system structure is that the disturbances are suppressed by increased amplification in the components which are located in the control circuit between the useful signal input and the component which is acted upon by the disturbances.

2. Physical Feasibility of Class of Structures Under Consideration

As was already stated elsewhere [13], and as follows from the results obtained in the preceding section of this paper, the system considered here can be carried out in reality, provided it belongs to the class of systems which remain stable when an unlimited increase of the amplification factors takes place. This question requires special consideration in this case: Stability or instability will depend upon the location of the poles of equation (5) or, which amounts to the same thing, upon the roots of the characteristic equation

$$\left[m^2 \prod_{i=1}^{4} D_i(p) + \prod_{i=1}^{4} R_i(p) K_2 K_4\right] D_2'(p) D_4'(p) = 0 \quad \text{at} \quad m \to 0. \tag{7}$$

In order that all roots of (7) shall be on the left from the imaginary axis on the plane of roots, it is necessary and sufficient that the roots of each of the factors in (7) be on the left from the imaginary axis on the roots plane at $m \to 0$. But since the roots of the equation $D'_2(p)D'_4(p) = 0$ do not depend upon m, these expressions can be left out from consideration. And we will assume that they are on the left of the imaginary axis. Thus, the stability will be determined by the roots of the following equation:

$$m^2 \prod_{i=1}^{4} D_i(p) + K_2 K_4 \prod_{i=1}^{4} R_i(p) = 0 \quad \text{at} \quad m \to 0. \tag{8}$$

It is known [10] that the roots of equation (8) will be located on the left from the imaginary axis, provided the following conditions are satisfied.

Let us designate the degree of the polynomial $\prod_{i=1}^{4} D_i(p)$ by N_2, and that of the polynomial $\prod_{i=1}^{4} R_i(p)$ by N_1. Then the conditions of stability will be as follows:

a) $N_2 - N_1 \leq 2$;

b) the equation $\prod_{i=1}^{4} R_i(p) K_2 K_4 = 0$ must satisfy the stability conditions;

c) depending upon whether $N_2 - N_1$ is equal two or one, a certain definite relationship must be valid between the coefficients of the polynomials $\prod_{i=1}^{4} D_i(p)$ and $K_2 K_4 \prod_{i=1}^{4} R_i(p)$.

Let us discuss the cases which are most difficult from the point of view of their being carried out in practice. Suppose that all components in the circuit of the system shown diagrammatically in Fig. 1 are elementary dynamic sections. For the sake of simplicity we shall consider the circuit of the system represented in Fig. 1 to be stable by design [14], and that all its components are of an aperiodic character.

If there is a total of N such components in the circuit, and v of them have large amplification factors, then equation (8) can be rewritten as follows:

$$m^{v}\prod_{i=1}^{N}(1+T_ip)+\prod_{i=1}^{n}K_i\prod_{i=n+v+1}^{N}K_i=0.$$

If we introduce the designation

$$\prod_{i=1}^{n}K_i\prod_{j=n+v+1}^{N}K_j=K_{\text{deg}},$$

then we obtain the following equation:

$$m^{v}\prod_{i=1}^{N}(1+T_ip)+K_{\text{deg}}=0. \tag{9}$$

It is obvious that equation (9) will satisfy the stability conditions only when $N \leq 2$. This is a trivial case which is of little interest. Let us investigate the possibility of stabilizing the system when $N > 2$.

From paper [10] it is known that to stabilize a system of the type represented by formula (9) it is necessary to introduce the derivatives at least from the (N-2)th to the first. Let us introduce into the system N−2 amplifiers whose amplification factors can be made sufficiently large. Let us correct these amplifiers by negative feedback paths having the transfer functions $\mu_i/(1+T_ip)$ (Fig. 2). Concerning the rest of the system we shall assume that among the N aperiodic components, there are α which are not directly subjected to the influence of disturbances, and that the amplification factors of these can be varied within wide limits. To simplify the computing operations, let us arrange the components which are directly acted upon by disturbances in two groups as shown in Fig. 2.

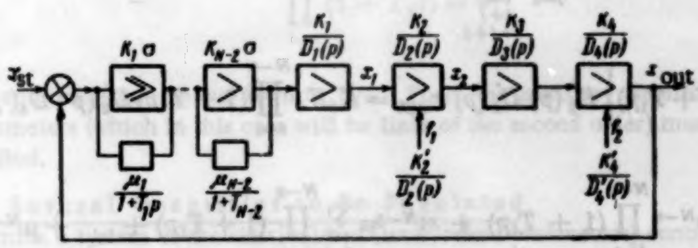


Fig. 2

For the first N−2 amplifiers – which use feedback – we have the following formulas:

$$x_{N-2}=\prod_{i=1}^{N-2}\frac{K_{i\sigma}(1+T_ip)}{T_ip+1+\mu_iK_{i\sigma}}, \tag{10}$$

$$x_1=\frac{K_1}{D_1(p)}x_{N-2}, \tag{11}$$

$$x_2=\frac{K_2}{D_2(p)}x_1+\frac{K_2'}{D_2'(p)}f_1, \tag{12}$$

$$x_3=\frac{K_3}{D_3(p)}x_2, \tag{13}$$

$$x_{\text{out}}=\frac{K_4}{D_4(p)}x_3+\frac{K_4'}{D_4'(p)}f_2. \tag{14}$$

Excluding x_1, x_2, x_3, and x_{N-2} from equations (10) to (14) we obtain

$$\left\{\prod_{i=1}^{4} D_i(p)\prod_{i=1}^{N-2}(T_ip+1+K_{i\sigma}\mu_i)\,D_2'(p)\,D_4'(p)+\right.$$

$$\left.+\prod_{i=1}^{4}K_i\prod_{i=1}^{N-2}K_{i\sigma}(1+T_ip)\,D_2'(p)\,D_4'(p)\right\}x_{\text{out}}=$$

$$=\prod_{i=1}^{4}K_i\prod_{i=1}^{N-2}K_{i\sigma}(1+T_ip)\,D_2'(p)\,D_4'(p)\;x_{st}+$$

$$+D_1(p)\,D_2(p)D_4'(p)\prod_{i=1}^{N-2}(T_ip+1+K_{i\sigma}\mu_i)\,K_4K_3K_2'f_1+$$

$$+\prod_{i=1}^{4}D_i(p)D_2'(p)\prod_{i=1}^{N-2}(T_ip+1+K_{1\sigma}\mu_1)\,f_2. \tag{15}$$

We divide both the right and left side of equation (15) by $\prod_{i=1}^{N-2}K_{i\sigma}K_1K_3$, and assume that the quantities $K_{i\sigma}$, K_1, and K_3 are all of the same order of magnitude. Then, designating $\frac{1}{K_{i\sigma}}=\frac{1}{K_1}=\frac{1}{K_3}=m$, after a few elementary operations we obtain the following expression:

$$\left\{m^2\prod_{i=1}^{4}D_i(p)\left[m^{N-2}\prod_{i=1}^{N-2}(1+T_ip)+m^{N-3}\mu\sum_{j=1}\prod_{\substack{i=1\\i\neq j}}^{N-2}(1+T_ip)+\right.\right.$$

$$\left.+m^{N-4}\mu^2\sum_{j=1}\prod_{\substack{i=1\\i\neq j\\i\neq j+1}}^{N-2}(1+T_ip)+\ldots+\mu^{N-2}\right]+$$

$$\left.+K_2K_4\prod_{i=1}^{N-2}(1+T_ip)\right\}D_2'(p)\,D_4'(p)\,x_{\text{out}}=K_2K_4\prod_{i=1}^{N-2}(1+T_ip)D_2'(p)\,D_4(p)x_{st}+$$

$$+m\left[m^{N-2}\prod_{i=1}^{N-2}(1+T_ip)+m^{N-3}\mu\sum_{j=1}\prod_{\substack{i=1\\i\neq j}}^{N-2}(1+T_ip)+\ldots+\mu^{N-2}\right]\times$$

$$\times K_4K_2'D_1(p)\,D_2(p)\,D_4'(p)\,f_1+m^2\left[m^{N-2}\prod_{i=1}^{N-2}(1+T_ip)+m^{N-3}\mu\times\right.$$

$$\left.\times\sum_{j=1}\prod_{\substack{i=1\\i\neq j}}^{N-2}(1+T_ip)+\ldots+\mu^{N-2}\right]\prod_{i=1}^{4}D_i(p)\,D_2'(p)\,f_2. \tag{16}$$

It is evident from equation (16) that the magnitude of the amplification factors of the component amplifiers using feedback has no effect upon reproduction of the disturbances: the disturbances will neither be more amplified nor more suppressed by changes of the amplification factors of these components; the only effect of these components upon the disturbances consists in changing their amplitudes by a factor of μ^{N-2}. When $\mu < 1$, the disturbances are correspondingly decreased. Substantially, however, like before, the disturbances are suppressed by a proper choice of amplification factors in the components which are not directly subjected to the action of the disturbances and are not provided with feedbacks.

Let us now examine the left side of equation (16) which when equated to zero becomes the characteristic equation of the system and determines the stability of the system, and hence the feasibility of its realization. We thus have

$$m^N \prod_{i=1}^{N-2}(1+T_i p)\prod_{i=1}^{4} D_i(p) + m^{N-1}\mu \sum_{i=1}^{N-2}\prod_{\substack{i=1\\ i\neq j}}^{N-2}(1+T_i p)\prod_{i=1}^{4} D_i(p) +$$

$$+ \ldots + m^2\mu^{N-2}\prod_{i=1}^{4} D_i(p) + K_2K_4\prod_{i=1}^{N-2}(1+T_i p) = 0. \tag{17}$$

The difference in degree of two consecutive polynomials in (17), except for the two last ones, is one, and the difference in degree of the last two polynomials is two, since the degree of the polynomial $\prod_{i=1}^{4} D_i(p)$ is equal 4 according to the conditions of the problem setup (because the case of N aperiodic circuit sections is considered). Thus, the structural conditions of stability are satisfied [10]. The degeneracy equation in the case under consideration is

$$K_2K_4\prod_{i=1}^{N-2}(1+T_i p) = 0, \tag{18}$$

which always satisfies stability conditions. Consequently, according to paper [10], for stability (feasibility) of the system it is necessary and sufficient that an auxiliary equation of the third kind [10] satisfy the stability conditions. And this can be secured by proper choices of values for the parameters T_i and for the amplification factors K_2K_4.

Thus the possibility of realization of the described structures has been proven. In the case just considered there were additionally N−2 amplifiers with large amplification factors introduced into the system. If N−1 amplifiers are introduced, instead of N−2 amplifiers, then an auxiliary equation of the first kind [10] will be obtained, which in the case under consideration satisfies the stability requirements always, because it is reduced to the form [10]

$$\prod_{i=1}^{n}(1+T_i q) = 0. \tag{19}$$

The number of amplifiers can be cut down to N/2. Then, the auxiliary equation will be of the second kind, and the feedback parameters (which in this case will be links of the second order) must be so chosen that the auxiliary equation will be satisfied.

3. System with Several Quantities to Be Regulated

Let us now examine a system of multiarticulated (interlocked parameters) control: it is assumed that in the controlled object there is not one quantity to be regulated, but several quantities all connected with each other through their serving to control the same object. Let us examine the case when each disturbance acts directly upon the object being controlled. It will be possible to extend the results of examination of this case to any other case without great difficulty.

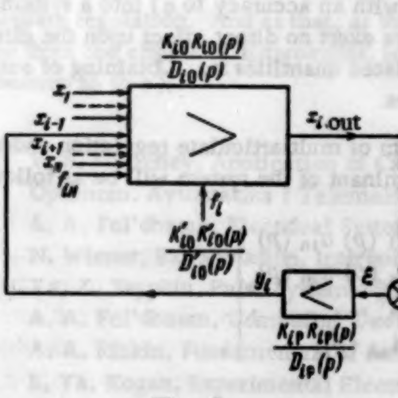


Fig. 3

In Fig. 3 we have presented the structure of a system for the first of the several regulated quantities in the case when there is a separate perturbation acting on each of the regulated quantities.

Let us write down the transfer function for the ith regulated quantity using the symbols and designations shown in Fig. 3:

$$x_i\ \text{out} = \frac{K_{i0}R_{i0}(p)}{D_{i0}(p)}\,y_i - \frac{K_{i0}R_{i0}(p)}{D_{i0}(p)}\sum_{\substack{v=1\\ v\neq 1}}^{n}\alpha_{iv}(p)\,x_v + \frac{K_{i0}R_{i0}(p)}{D_{i0}(p)}\,f_{in} + \frac{K'_{i0}R'_{i0}(p)}{D'_{i0}(p)}\,f_i, \tag{20}$$

$$\xi = x_{i\,st} - x_{i\,out}, \tag{21}$$

$$y_i = \frac{K_{ip}R_{ip}(p)}{D_{ip}(p)} x_{i\,st} - \frac{K_{ip}R_{ip}(p)}{D_{ip}(p)} x_{i\,out}. \tag{22}$$

Excluding y_i and ξ from equations (20)-(22), we obtain, after the necessary transformations have been carried out, the following expression:

$$[D_{io}(p)D_{ip}(p) + K_{ip}R_{io}(p)]\,D'_{io}(p)\,x_{out} + D'_{io}(p)\,D_{ip}(p)\,K_{io}R_{io}(p) \times \sum_{\substack{v=1\\ v\neq i}}^{n} \alpha_{iv}(p)\,x_v = K_{ip}K_{io}R_{io}(p)\,D_{io}(p)\,x_{st} + D_{io}(p)\,D_{ip}(p)\,K_iR'_{io}(p)\,f_i + K_{io}R_{io}(p)\,D_{io}(p) f_{iH}. \tag{23}$$

In the above equations the subscript "o" is used to distinguish those members which pertain to the controlled object; the letter p (= Russian r) when used as a subscript designates that the quantity to which it is a subscript pertains to a regulated quantity; f_i is a disturbance applied to the ith regulated quantity; f_{iH}—the load in the ith regulated quantity; n—the total number of regulated quantities; and $\alpha_{iv}(p)$—an operator which determines the connection between the ith and the vth regulated systems.

The complete system of all equations for the multiarticulate system of regulation is obtained by giving i in equation (23) all values from i = 1 to i = n.

We divide equation (23) by K_{ip} and substitute m for $1/K_{ip}$. Then we obtain the following:

$$[mD_{io}(p)\,D_{ip}(p) + K_{io}R_{io}(p)]\,D_{io}(p)\,x_{out} + mD'_{io}D_{ip}(p)\,K_{io}R_{io}(p)\sum_{\substack{v=1\\ v\neq i}}^{n} \alpha_{iv}(p)\,x_v = K_{io}R_{io}(p)\,D'_{io}(p)\,x_{st} + mD_{io}(p)\,D_{ip}(p)\,K'_iR'_{io}(p)\,f_i + mK_{io}R_{io}D_{ip}(p)\,D'_{io}(p)\,f_{iH}. \tag{24}$$

It is evident from (24) that if the system remains stable at $m \to 0$, that is, at $K_{ip} \to \infty$, then

$$\lim_{m\to 0} x_{out} = x_{i\,st}. \tag{25}$$

Thus, in this case also a disturbance in the ith circuit loop can be suppressed by an increase of the amplification factor of the ith regulator. It follows from (25) that together with the disturbance also the load interaction f_{iH} is removed, and thus the multiarticulate (interlocked) system is resolved (with an accuracy to ε) into a system of n independent regulation systems. The disturbances in the other circuit loops exert no direct effect upon the circuit loop under consideration, only an indirect effect by way of the other regulated quantities x_v. Obtaining of autonomy in this case simultaneously removes the influence of "foreign" disturbances.

It remains to examine the question of stability of the resulting system of multiarticulate regulation under the condition that all K_{ip} at i = 1, . . . , n tend toward infinity. The determinant of the system will be as follows:

$$\Delta = \begin{vmatrix} A_{11}(p),\ B_1(p)\,\alpha_{12}(p),\ B_1(p)\,\alpha_{13},\ldots,\ B_1(p)\,\alpha_{1n}(p) \\ B_2(p)\,\alpha_{21}(p)\ A_{22}(p),\ B_2(p)\,\alpha_{23},\ldots,\ B_2(p)\,\alpha_{2n}(p) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ B_n(p)\,\alpha_{n1}(p),\ B_n(p)\,\alpha_{n2}(p),\ldots,\ A_{nn}(p), \end{vmatrix}, \tag{26}$$

where

$$A_{ii}(p) = [mD_{io}(p)\,D_{ip}(p) + K_{io}R_{io}(p)]\,D'_{io}(p),$$

$$B_i(p) = mD'_{io}(p)\,K_{io}R_{io}(p).$$

The characteristic equation which determines whether the entire system is stable or not is as follows:

$$\Delta = 0. \tag{27}$$

Equations of the type of (27) were encountered already in [10]. It was shown there that, if the systems for every regulated quantity remain stable at $m \to 0$, then also the complete system in its entirety will be stable, provided that the degeneracy equation of the complete multiarticulate system satisfies the stability conditions and that auxiliary equations of the first, second, or third kind too satisfy the stability conditions [10].

It follows from the results of the preceding section, that it is possible to stabilize every separate system by introduction of additional amplifiers with sufficiently high amplification factors provided with negative feedbacks having transfer functions of the type

$$\frac{\mu_i}{1 + T_i p} \quad \text{or} \quad \frac{\mu_i}{ap^2 + bp + 1}.$$

It is not difficult to see that in this case the stability of the degenerate system will fully depend upon the parameters of the components which are included in the feedback lines of the amplifiers. Hence, stability of the degenerate system can always be secured.

For the case when the auxiliary equation is of the first kind, it satisfies the stability conditions automatically; if the auxiliary equation is of the second or third kind, it is always possible to secure satisfaction of the stability conditions by the auxiliary equation by a suitable choice of the parameters in the regulators and in the amplifier feedbacks.

Thus, in this case it is possible to secure stability, and hence feasibility of real applications of the described system.

SUMMARY

On the basis of the investigation carried out by us, the following conclusions can be made:

1. If in a closed dynamic system not all components are subject to the influence of disturbances from the outside, then the suppression of the disturbances can be achieved by an increase of the amplification factors of those components which are not directly subject to the action of the disturbances. And this is brought about by the fact that the disturbances from any given component are suppressed by the amplifications in the system components preceding that particular component, starting the count from the input to the system of the useful, or control, signal as the beginning.

2. It is feasible to turn such systems into realities, if their structure is of the class which remains stable when an unlimited increase of the amplification factors takes place.

3. The obtained results cover not only systems with a single regulated quantity, but also systems of multiarticulate regulation. And at that, as should be expected [8], the multiarticulate system, simultaneously with being freed from the effects of disturbances, is also dissociated into separate autonomous systems (with an accuracy of autonomy up to ε).

LITERATURE CITED

1. V. S. Pugachev, Application of Canonical Resolution of Random Functions to Determination of a Linear System Optimum. Avtomatika i Telemekhanika, 17, No. 6 (1956).
2. A. A. Fel'dbaum, Electrical System of Automatic Regulation. Oborongiz. State Defence Press (1954).
3. N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series, John Wiley (1949).
4. Ya. Z. Tsypkin, Pulse Systems. State Phys. and Math. Press (1958).
5. A. A. Fel'dbaum, Computing Devices of Automatic Systems. State Phys. and Math. Press (1960).
6. A. A. Rizkin, Fundamentals of Amplification Circuit Theory. Soviet Radio Press (1954).
7. B. Ya. Kogan, Experimental Electronic Models and Their Application for Testing of Automatic Regulation Systems. State Phys. and Math. Press (1959).
8. M. V. Meerov, Principles for Design of Automatic Regulation Systems with Small Stationary Errors. Avtomatika i Telemekhanika, 10, No. 2 (1949).

9. M. V. Meerov, On Automatic Regulation Systems Stable with Amplifications as High as Desired. Avtomatika i Telemekhanika, 7, No. 4 (1947).

10. M. V. Meerov, Synthesis of Structure of Automatic Regulation with High Accuracy. State Phys. and Math. Press (1959).

11. M. V. Meerov, Systems of Multiarticulate Regulation Stable at Small Stationary Errors. Proc. of 2nd All-Union Conference on Automatic Control (1955) vol. 2.

12. M. V. Meerov, On Autonomy of Multiarticulate Systems Which Remain Stable with Unlimited Increase of Established Accuracy. Avtomatika i Telemekhanika, 17, No. 5 (1956).

13. M. V. Meerov, On Stability Against Disturbances of a Certain Class of Dynamic Systems. Doklady Akad. Nauk SSSR, 139, No. 4 (1961).

14. M. A. Aizerman, Certain Structural Conditions for Stability of Automatic Regulation Systems. Avtomatika i Telemekhanika, 10, No. 2 (1948).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

THEORY OF CONTROL SYSTEMS WITH LIMITED-SPEED SERVOMECHANISMS*

L. S. Gol'dfarb

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1324-1332, October, 1961

Original article submitted March 15, 1961

A paper [1] concerning investigations of the stability and self-oscillations of automatic control systems having a nonlinear element – a limited-speed servomotor with rigid negative feedback – was submitted to the IFAC First International Congress on Automatic Control. The analysis was performed by using the phase plane method as well as the describing-function approach. In this, the nonlinear element was considered as an elementary factor, which rendered the describing-function approach more complicated and reduced its effectiveness and also impaired the accuracy and clarity of the investigation results.

In the present article, the detailed structure of a nonlinear element is considered, which makes it possible to simplify and elucidate the investigations, to obtain some new results, and to consider more complex cases of practical importance.

Paper [1] is concerned with an automatic control system (Fig. 1) which consists of two inertial elements and a limited-speed servomotor, which is encompassed by rigid negative feedback. The input quantity E of the nonlinear element is related to its output F by the relationships

$$E = F \quad \text{for} \quad \left|\frac{dE}{dt}\right| < \dot{F}_m,$$

$$\frac{dF}{dt} = \dot{F}_m \quad \text{for} \quad F < E, \tag{1}$$

$$\frac{dF}{dt} = -\dot{F}_m \quad \text{for} \quad F > E,$$

where $\dot{F}_m$ is positive and corresponds to the maximum rate of change in F that the servomotor can secure.

In [1], the system's motion is investigated by using the phase plane method and the describing-function approach. The article provides a comparison between the values of self-oscillation periods for equal time constants of the inertial elements in dependence on their gains SM. These values were calculated by using both methods. It is stated that the describing-function approach secures a satisfactory accuracy.

The method used by the authors of paper [1] for the mathematical description of a limited-speed servomotor with rigid feedback leads to the fact that the nonlinear element's equivalent gain vector (1) contains an imaginary component. This is connected with the fact that the element under consideration includes a nonlinear element with a "limited power" characteristic (NÉ IV in [2]) as well as a linear element – an integrating factor.

The complexity of the equivalent gain complicates calculations and makes them less clear.

This drawback can be avoided by separating the nonlinear element in the manner shown in Fig. 2. By considering the nonlinear element's structure in detail, it is possible greatly to simplify the calculations and to make them clear in the highest degree for the case considered in [1] as well as for systems with more complex servomechanism dynamics, whereby the investigation method can be extended to a larger number of practical problems.

* The article was prepared for publication by E. B. Pasternak.

The block diagram in Fig. 2 apparently provides a more accurate description of the properties of a system with a limited-speed servomotor that is encompassed by negative feedback.

The servomotor is characterized by the maximum rate $\dot{F}_m$ and by the width $\underline{a}$ of the linear zone. The gain in the linear zone is

$$\mu = \frac{\dot{F}_m}{a}. \tag{2}$$

The mathematical description of the rigid feedback loop W_2, which corresponds to NF in [1], coincides with (1) if μ tends to infinity.

Actually, for $-\dot{F}_m < \dot{F} < \dot{F}_m$, the transfer function of the loop

$$W_2 = \frac{\frac{\mu}{p}}{\frac{\mu}{p}+1} = \left(\frac{1}{\mu}p+1\right)^{-1}$$

tends to unity with an increase in μ. In this, F = E in the limit. It is obvious from Fig. 2 that the other conditions are satisfied.

The consideration of the degenerate case where $\mu = \infty$ enabled the authors of paper [1] to reduce the analysis of a third-order system to the analysis of a second-order system, which made it possible to consider its motion in the phase plane. However, by determining the equivalent gain according to (1), they also limited the describing-function approach to this case of maximum idealization, which apparently was not necessary.

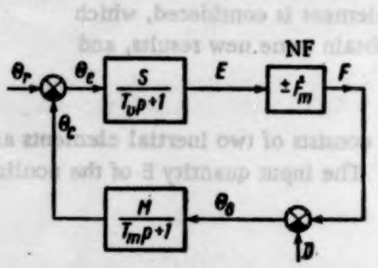


Fig. 1. Block diagram of the nonlinear system of [1].

In the present paper, such limitation is not imposed, and the limiting case is considered separately at the end of the article.

The transfer function of the linear portion of the system is

$$W_l\,(p) = \frac{1}{p}\,[1 + W_1\,(p)], \tag{3}$$

where $W_1(p) = SM/[(T_m p + 1)(T_v p + 1)]$ is the transfer function of the two series-coupled inertial elements, and T_m and T_v are positive real numbers. The corresponding complex gain is

$$W_1\,(j\omega) = \frac{SM}{(1 + j\omega T_m)(1 + j\omega T_v)}.$$

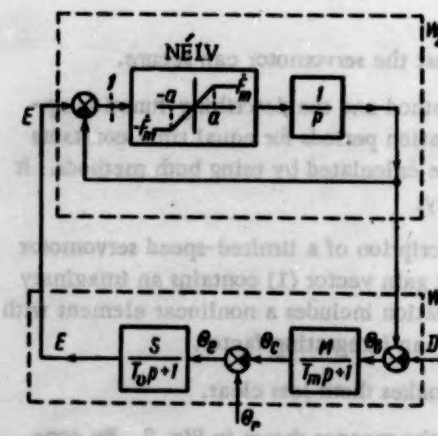


Fig. 2. Block diagram of the control system.

Let us consider the following parameters:

$$\Omega = \omega\sqrt{T_m T_v}, \quad 2d = \frac{T_m + T_v}{\sqrt{T_m T_v}}, \quad d \geqslant 1.$$

Then,

$$W_1\,(j\Omega) = \frac{SM}{(1-\Omega^2) + j2d\Omega}, \tag{4}$$

$$\mathrm{Re}\, W_1 = \frac{(1-\Omega^2)\,SM}{(1-\Omega^2)^2 + 4d^2\Omega^2}, \quad \mathrm{Im}\, W_1 = -\frac{2dSM\Omega}{(1-\Omega^2)^2 + 4d^2\Omega^2}, \tag{5}$$

$$W_l\ (j\Omega) = \frac{SM\sqrt{T_m T_v}}{j\Omega} W_0\,(j\Omega), \tag{6}$$

where

$$W_0\,(j\Omega) = -\frac{1}{SM} + \frac{1}{(1-\Omega^2)+j2d\Omega} \tag{7}$$

is the normalized complex gain, the hodograph of which (Fig. 3) represents the hodograph of the normalized complex gain $\frac{1}{SM}W_1\,(j\Omega) = \frac{1}{(1-\Omega^2)+j2d\Omega}$ of the series-coupling of two inertial elements, which is shifted by $\frac{1}{SM}$ to the right of the imaginary axis.

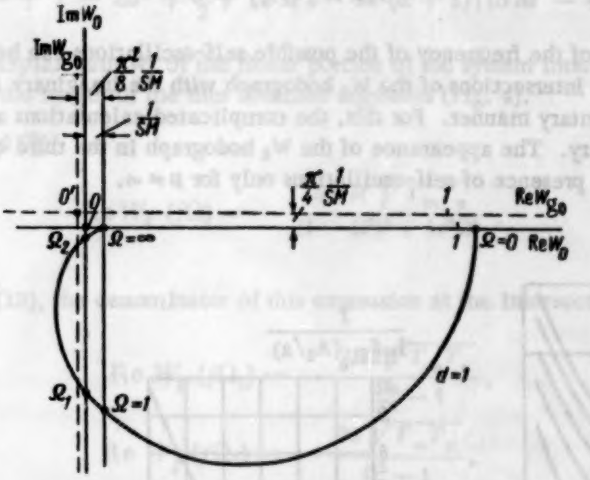


Fig. 3. Hodographs of $W_0(j\Omega)$ and $W_{g0}(j\Omega)$.

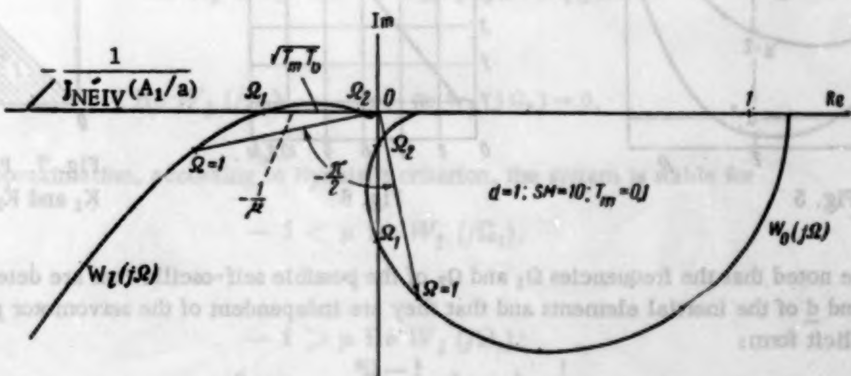


Fig. 4. Construction of the W_l $(j\Omega)$ and $-\frac{1}{J_{NEIV}}$ hodographs.

The vectors $W_l(j\Omega)$ are rotated through $-\pi/2$ with respect to the vectors $W_0(j\Omega)$, and

$$\operatorname{Re} W_l\ (j\Omega) = \frac{SM\sqrt{T_m T_v}}{\Omega}\ \operatorname{Im}\ W_0\ (j\Omega), \tag{8}$$

$$\operatorname{Im} W_l\,(j\Omega) = -\frac{SM\sqrt{T_m T_v}}{\Omega} \operatorname{Re} W_0\ (j\Omega). \tag{9}$$

The vector of the nonlinear element's equivalent gain $J_{NÉIV}$[2] contains only a real component. Thus, the possible self-oscillation regimes of the system (Fig. 4)

$$W_l\,(j\Omega) = -\frac{1}{J_{NÉIV}\left(\frac{A_1}{a}\right)} \quad (\Omega \neq \infty) \tag{10}$$

correspond to the roots of the equation

$$\text{Im}\; W_l\,(j\Omega) = 0, \tag{11}$$

whence

$$\text{Re}\; W_0\,(j\Omega) = 0. \tag{12}$$

Thus, the determination of the frequency of the possible self-oscillations can be reduced to the determination of frequencies Ω_1 and Ω_2 at the intersections of the W_0 hodograph with the imaginary axis. The frequencies Ω_1 and Ω_2 are determined in an elementary manner. For this, the complicated calculations and graphical constructions presented in [1] are not necessary. The appearance of the W_0 hodograph in the third quadrant of the complex plane is a necessary condition for the presence of self-oscillations only for $\mu \neq \infty$.

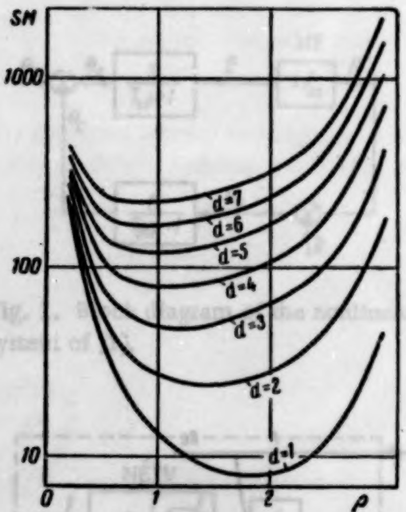


Fig. 5

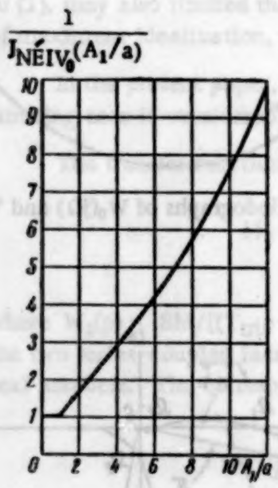


Fig. 6

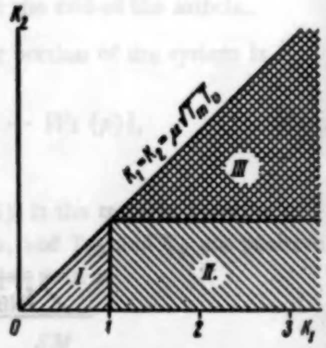


Fig. 7. Plane of the parameters K_1 and K_2 for $SM \geq SM_{cr}$.

It should be noted that the frequencies Ω_1 and Ω_2 of the possible self-oscillations are determined only by the parameters SM and d of the inertial elements and that they are independent of the servomotor parameters. Let us write (12) in explicit form:

$$\frac{1}{SM} + \frac{1-\Omega^2}{(1-\Omega^2)^2 + 4d^2\Omega^2} = 0,$$

whence

$$(1-\Omega^2)^2 + 4d^2\Omega^2 + SM\,(1-\Omega^2) = 0. \tag{13}$$

Let us introduce the parameter $\rho = \frac{\pi}{\Omega}$, which was considered in [1]; it is obvious from Fig. 3 that $\infty > \Omega > 1$. Consequently, $0 < \rho < \pi$.

Dependence (13) is represented graphically in Fig. 5. The minimum value of SM that secures the appearance of self-oscillations is given by

$$SM_{cr} = 4d\,(d + 1). \tag{14}$$

In this,

$$\Omega_1^2 = \Omega_2^2 = 2d + 1.$$

If SM > SM_{cr}, there are two different intersection frequencies $1 < \Omega_1 < 2d + 1 < \Omega_2$:

$$\Omega_1 = \sqrt{\frac{SM}{2} + 1 - 2d^2 - \frac{1}{2}\sqrt{[SM - 4d\,(d+1)]\,[SM - 4d\,(d-1)]}}, \tag{15}$$

$$\Omega_2 = \sqrt{\frac{SM}{2} + 1 - 2d^2 + \frac{1}{2}\sqrt{[SM - 4d\,(d+1)]\,[SM - 4d\,(d-1)]}}. \tag{16}$$

The hodograph of the complex gain W_l of the linear portion of the system intersects the real axis at $\Omega = \Omega_1$ and $\Omega = \Omega_2$. Let us determine the length of the thus obtained segments (Fig. 4).

According to (5), (7), and (8),

$$\operatorname{Re} W_l\,(j\Omega) = -\frac{2dSM\sqrt{T_m T_v}}{(1-\Omega^2)^2 + 4d^2\Omega^2}.$$

If we take into account (13), the denominator of this expression at the intersection points is equal to $SM(\Omega_{1,2}^2 - 1)$, whence

$$\operatorname{Re} W_l\,(j\Omega_1) = -\frac{2d\sqrt{T_m T_v}}{\Omega_1^2 - 1},$$

$$\operatorname{Re} W_l\,(j\Omega_2) = -\frac{2d\sqrt{T_m T_v}}{\Omega_2^2 - 1}, \tag{17}$$

while

$$\operatorname{Re} W_l\,(j\Omega_1) \leqslant -\sqrt{T_m T_v} \leqslant \operatorname{Re} W_l(j\Omega_2). \tag{18}$$

If SM increases,

$$\operatorname{Re} W_l\,(j\Omega_1) \to -\infty, \quad \operatorname{Re} W_l\,(j\,\Omega_2) \to 0.$$

In the linear approximation, according to Nyquist's criterion, the system is stable for

$$-1 < \mu \operatorname{Re} W_l\,(j\Omega_1),$$

and for

$$-1 > \mu \operatorname{Re} W_l\,(j\Omega_2). \tag{19}$$

The meaning of (19) is obvious. The transfer function $(\frac{1}{\mu}p + 1)^{-1}$ of the servomotor loop corresponds to an inertial factor with the time constant $1/\mu$ and a unit gain. In the linear approximation, the system is equivalent to a system consisting of three inertial elements, which is stable for relatively small as well as relatively large time constant values of one of these elements.

With an increase in μ, the system becomes equivalent to a system consisting of two inertial elements, which is stable for any parameter values.

In the presence of limitations, the system loses this property and becomes in this case a self-oscillation system with a severe excitation regime. The stability of the system's equilibrium position "in the small" does not ensure its operating ability, which once again emphasizes the necessity of taking into account the nonlinearities of the characteristics in determining the stability of even such simple systems.

Let us write (10) in explicit form by using (17):

$$-\frac{2d\sqrt{T_{.m}T_v}}{\Omega^2-1} = -\frac{1}{J_{N\acute{E}IV}} \qquad (\Omega=\Omega_1,\ \ \Omega=\Omega_2),$$

$$J_{N\acute{E}IV} = \frac{F_m}{a}J_{N\acute{E}IV_0} \qquad (0<J_{N\acute{E}IV_0}<1).$$

According to [2],

$$J_{N\acute{E}IV_0} = \frac{4}{\pi}\left[\frac{1}{2}\alpha - \frac{\sin 2\alpha}{4} + \frac{\cos\alpha}{\frac{A_1}{a}}\right],$$

where $\alpha = \arcsin\frac{a}{A_1}$; A_1 is the amplitude of self-oscillations at the input of the nonlinear element. The hodograph $-\frac{1}{J_{N\acute{E}IV}}$ corresponds to the $-\frac{1}{\mu} \geq -\frac{1}{J_{N\acute{E}IV}}$ section of the real axis (Fig. 4); the $\frac{1}{J_{N\acute{E}IV_0}\left(\frac{A_1}{a}\right)}$ curve is given in Fig. 6.

The equation of the system's self-oscillations is

$$K_{1,2} = \frac{2F_m d\sqrt{T_m T_v}}{a(\Omega_{1,2}^2-1)} = \frac{1}{J_{N\acute{E}IV_0}}, \qquad a \neq 0. \tag{20}$$

Three regular cases are possible (Fig. 7).

I. $K_2 < K_1 < 1$. As in the case where $SM < SM_{cr}$, the system is stable in the linear approximation and also if the speed limitation is taken into account. The hodograph of the complex gain of the system's linear portion intersects the real axis to the right of the point

$$-\frac{1}{\mu}.$$

II. $K_2 < 1 < K_1$. In this case, the hodograph of the complex gain of the system's linear portion intersects only once the hodograph of the nonlinear element's equivalent gain.

Fig. 8

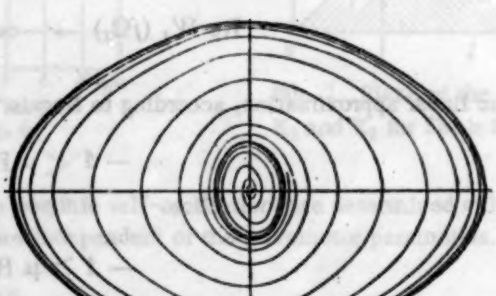


Fig. 9

The intersection point $K_1 = \frac{1}{J_{N\acute{E}IV_0}}$ corresponds to a stable limiting cycle [2]; the equilibrium position in the linear approximation is unstable. A self-oscillation regime with soft excitation is observed in the system. Figure 8 shows the phase pattern of the system's model for this regime, which was obtained by means of an MN-7 electronic simulator.

III. $1 < K_2 < K_1$. In this case, there are two intersection points, of which the point $K_1 = \frac{1}{J_{N\acute{E}IV_0}}$ corresponds to a stable limiting cycle, while the point $K_2 = \frac{1}{J_{N\acute{E}IV_0}}$ corresponds to an unstable limiting cycle. On the basis of (19),

the equilibrium position is stable. The system operates under the regime of severe excitation of self-oscillations. The corresponding phase pattern for the system's model is shown in Fig. 9. By using Fig. 7, the self-oscillation amplitude can readily be determined from Eq. (20).

In conclusion, we shall consider the particular case where a = 0, which was considered in [1]. For this, the system becomes a relay system. The transfer function of the linear portion

$$W_l\,(p) = \frac{1}{p} + \frac{SM}{p\,(T_m p + 1)\,(T_v p + 1)}$$

contains the term 1/p, where the order of the numerator is only by unity lower than the order of the denominator. In calculating such systems, the describing-function approach in its usual form can result in considerable errors, which can be avoided by taking into account the "jump" phenomenon [3].

A more accurate complex gain for the servomotor's feedback circuit can readily be determined analytically by taking into account the jump and the exact signal shape:

$$W_{\text{g.fb}} = \frac{1}{j\Omega}\Big(-\frac{\pi}{4}j + \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\Big) = \frac{\pi}{4j\Omega}\Big(\frac{\pi}{2} - j\Big). \tag{21}$$

The more accurate complex gain of the system's linear portion is

$$W_{\text{g.}l} = \frac{SM\sqrt{T_m T_v}}{j\Omega}W_{\text{g}0}, \tag{22}$$

where

$$W_{\text{g}0} = \frac{\pi}{4}\left(\frac{\pi}{2} - j\right)\frac{1}{SM} + \frac{1}{(1-\Omega^2) + j2d\Omega} = \left(\frac{\pi^2}{8} - 1\right)\frac{1}{SM} - j\,\frac{\pi}{4}\,\frac{1}{SM} + W_0. \tag{23}$$

Thus, the consideration of the jump leads to a somewhat greater ($\frac{\pi^2}{8}\,\frac{1}{SM}$ instead of $\frac{1}{SM}$) shift of the normalized hodograph $\frac{1}{SM}\,W_1(j\Omega)$ to the right of the imaginary axis and to a downward $\frac{\pi}{4}\,\frac{1}{SM}$ shift with respect to the real axis (Fig. 3, dashed line).

Since, in this case,

$$J'_{\text{NE}} = \lim_{a\to 0} J'_{\text{NEIV}} = \frac{4}{\pi}\,\frac{\dot{F}_m}{A_1},$$

the frequency and the amplitude of the limiting cycles can readily be determined analytically from equations similar to (13) and (20):

$$(1-\Omega^2)^2 + 4d^2\Omega^2 - \frac{8}{\pi^2}SM\,(\Omega^2 - 1) = 0, \tag{24}$$

$$A_1 = \frac{\dot{F}_m\sqrt{T_m T_v}}{\Omega}\left(1 + \frac{\pi d\Omega}{\Omega^2 - 1}\right). \tag{25}$$

In [1], the describing-function approach was used for determining only the half-period $\rho = \frac{\pi}{\Omega}$ for d = 1 and for six values of SM.

The diagrams of the half-period and amplitude of self-oscillations in dependence on the system's parameters were plotted (Figs. 10 and 11) on the basis of (24) and (25).

For the sake of comparison, Fig. 10 also shows the results from [1], which were obtained by using the phase plane method and the describing-function approach.

It should be noted that the exact results for the limiting case can be obtained in a much simpler way by using the frequency method of investigation relay systems [4] instead of using the phase plane method.

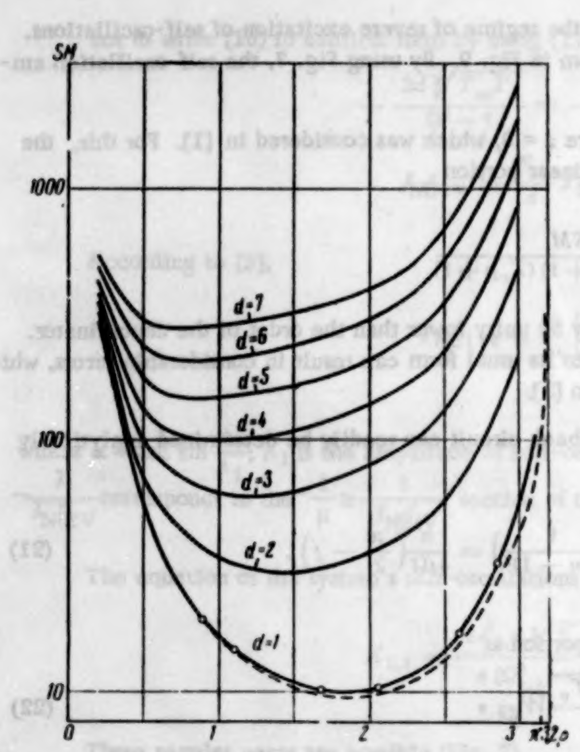


Fig. 10. ------) Results of the exact solution [1]; ○) results obtained in [1] by using the describing-function approach.

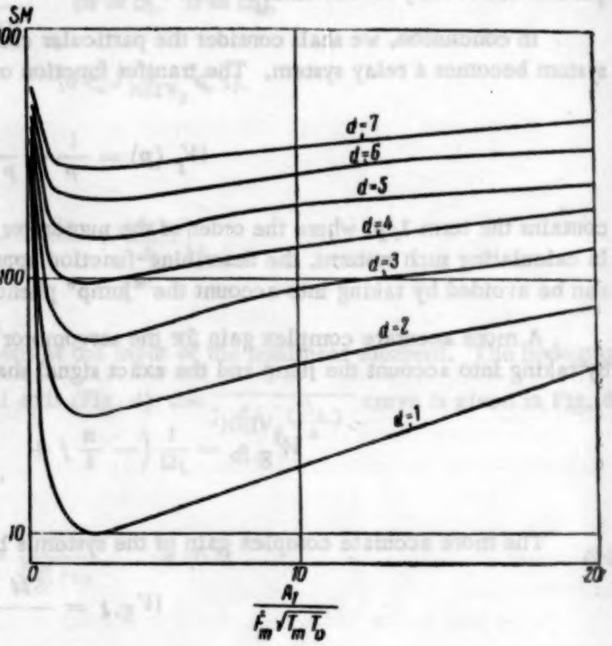


Fig. 11

LITERATURE CITED

1. I. Takaasi, J. G. Ziegler, and N. B. Nichols, "Control processes in the case of speed limitations," Report to the IFAC First International Congress on Automatic Control [Russian translation] (Moscow, 1960).
2. L. S. Gol'dfarb, "On certain nonlinearities in control systems," Avtomatika i Telemekhanika, 8, No. 5 (1947).
3. N. A. Korolev, "Approximate determination of self-oscillation parameters in relay systems with retarded feedback," Avtomatika i Telemekhanika, 20, No. 11 (1959).
4. Ya. Z. Tsypkin, Theory of Automatic Control Relay Systems [in Russian] (Gostekhizdat, 1955).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

AN ANALYSIS OF UNITARY-CODE AUTOMATIC SYSTEMS

Lü Ying-hsiang
(Moscow)
Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1333-1339, October, 1961
Original article submitted March 27, 1961

It is demonstrated that an investigation of unitary-code automatic systems of the type under study can be reduced to an investigation of analog systems that contain equivalent nonlinear elements.

A method is given for analyzing unitary-code automatic systems.

An analysis is made of the accuracy of a specific unitary-code system, and recommendations are given for increasing the accuracy of such a system.

With the development of mathematical machines digital computers are finding an ever wider application in automatic control and regulation systems.

Digital automatic systems in which digital data correspond not to the total values of the physical quantities but to their increments shall be called unitary-code automatic systems (UCAS). Such systems are finding the widest application in industry.

Notwithstanding the wide usage of UCAS, the problems of the dynamics in such systems have not as yet been adequately studied. The solution of the problem is complicated by the fact that the methods for analyzing conventional pulse systems are inapplicable to UCAS.

In this paper we shall demonstrate that the investigation of UCAS of a given type can be reduced to an investigation of continuous systems containing equivalent nonlinear elements. We shall estimate the accuracy of a specific unitary-code system and point out a method for increasing its accuracy.

The Equivalent Block Diagram

The block diagram of the UCAS is shown in Fig. 1, where 1, 5 are coding units, 2 is a reversive counter, 3 is a decoder, and 4 is the analog section of the system.

In this setup the coding units 1 and 5 convert analog data into unitary codes (Fig. 2), and therefore the signals U_y and U_x represent trains of pulses with an identical height and different time intervals between adjacent pulses.

The reversive counter with the carry circuit is a comparison block. The comparison operation in the counter consists of the following: the increments of the quantities are applied in the form of pulses to the reversive counter where the increment pulses U_y are added and the increment pulses U_x are subtracted. As a result of this the number indicated by the counter is equal to the difference $\Sigma U_y - \Sigma U_x$ at each instant of time.

The decoder converts the binary code (the outputs of the reversive counter triggers) into a voltage whose magnitude is proportional to the number of pulses $\Sigma U_y - \Sigma U_x$.

The block diagram shown in Fig. 1 can be converted to the form shown in Fig. 3, where $\Psi[y]$ and $\Psi[x]$ are equivalent nonlinear elements of a special type whose characteristics are shown in Fig. 4a; W(s) is the transfer function of the linear analog section of the system.

It is evident that the equivalent nonlinear elements $\Psi[y]$ and $\Psi[x]$ can be treated as a parallel connection of linear and nonlinear sections (Fig. 4b) [1]. Therefore the block diagram shown in Fig. 1 can be reduced to a different form (Fig. 5) when the conversion coefficient K_1 is taken into account.

Thus the investigation of unitary-code automatic systems can be reduced to an investigation of continuous systems containing equivalent nonlinear elements of a special type.

Estimating the Accuracy

We shall use k(t) to denote the pulse transient function for the limiting linear analog system. Then in accordance with Fig. 5 we can write

$$x(t) = \int_0^\infty [y(t-\tau) + \Phi_{an}(x, y, t-\tau)]\, k(\tau)\, d\tau. \quad (1)$$

For the system error we obtain the expression

$$\varepsilon(t) = \left\{ y(t) - \int_0^\infty y(t-\tau)\, k(\tau)\, d\tau \right\} + \int_0^\infty \Phi_{an}(x, y, t-\tau)\, k(\tau)\, d\tau. \quad (2)$$

The first term in the right side of Eq. (2), which determines the dynamic error $\varepsilon_d(t)$ of the system, can be computed in the conventional manner. The second term, which determines the error caused by level quantization, is defined in accordance with an upper tolerance.

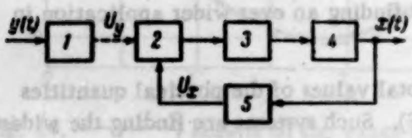


Fig. 1

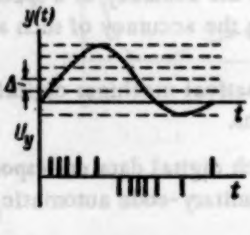


Fig. 2

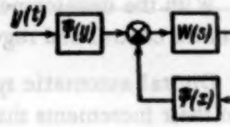


Fig. 3

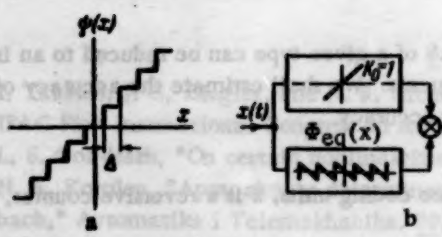


Fig. 4

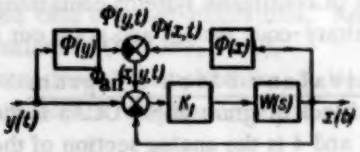


Fig. 5

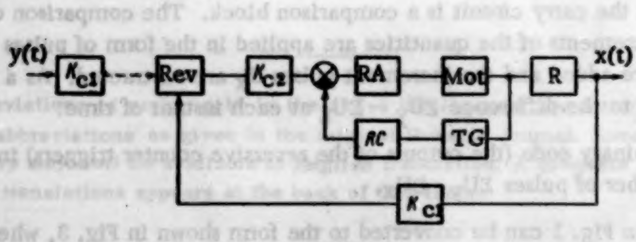


Fig. 6

From Fig. 5 it is evident that $\Phi_{an}(x, y, t) = \Phi(x, t) - \Phi(y, t)$. From this we obtain the expression for estimating the absolute value of the function $\Phi_{an}(x, y, t)$:

$$|\Phi_{an}(x, y, t)| \leqslant |\Phi(x, t)| + |\Phi(y, t)|.$$

But the difference between the absolute values of quantized and unquantized quantities cannot exceed one-half the quantization step $\Delta/2$; i.e., $|\Phi(x, t)| \le \Delta/2$; $|\Phi(y, t)| \le \Delta/2$ (cf. Figs. 4a and 4b). As a result we obtain the relationship

$$|\Phi_{\mathrm{an}}(x, y, t)| \leqslant |\Phi(x, t)| + |\Phi(y, t)| \leqslant \Delta.$$

Therefore the maximum error caused by level quantization is estimated as

$$|\varepsilon_{\mathrm{an}}(t)_{\max}| \leqslant \int_0^t |\Phi_{\mathrm{an}}(x, y, t-\tau)|\,|k(\tau)|\,d\tau \leqslant \Delta \int_0^t |k(\tau)|\,d\tau.$$

In a steady-state mode we shall have the inequality

$$|\varepsilon_{\mathrm{an}}(\infty)_{\max}| \leqslant \Delta \int_0^\infty |k(\tau)|\,d\tau. \tag{3}$$

From this it follows that if the pulse transient function k(t) is non-negative, then the maximum error caused by level quantization will not exceed a value equal to the value of the product of the quantization step and the value of the transient function at the same instant; the maximum steady-state error produced by level quantization will not exceed the value of Δ.

In the case where the pulse transient function changes sign we have

$$|\varepsilon_{\mathrm{an}}(\infty)_{\max}| \leqslant \eta\Delta \cdot \quad (\eta > 1). \tag{4}$$

We shall use ε_a to denote the error which arises during the process of preparing the program (i.e., the approximation error during the preparation of the information). Then we shall obtain the following expressions for the maximum total error of the UCAS:

$$\begin{gathered}|\varepsilon_\Sigma(t)_{\max}| \leqslant |\varepsilon_d(t)| + |\varepsilon_{\mathrm{an}}(t)_{\max}| + |\varepsilon_a|,\\ |\varepsilon_\Sigma(\infty)_{\max}| \leqslant |\varepsilon_d(\infty)| + |\varepsilon_{\mathrm{an}}(\infty)_{\max}| + |\varepsilon_a|.\end{gathered} \tag{5}$$

Analysis of a Certain Unitary-Code System

We shall study a unitary-code system for program control of metal-cutting machine tools; its block diagram is shown in Fig. 6 [3]. The transfer function for the analog section of the open-loop system is given by the formula

$$NW(s) = K_{\mathrm{an}} W_1(s) W_2(s), \tag{6}$$

where

$$K_{\mathrm{an}} = \frac{N}{K_{TG} K_{RC}}, \quad W_1(s) = \frac{1 + T_3 s}{s^2},$$

$$W_2(s) = \frac{\dfrac{K_{RA} K_{Mo} K_{TG} K_{RC} s}{(1+T_1 s)(1+T_2 s)(1+T_3 s)}}{1 + \dfrac{K_{RA} K_{Mo} K_{TG} K_{RC} s}{(1+T_1 s)(1+T_2 s)(1+T_3 s)}}.$$

Here N = 1/47 is the transfer number of the reducer. $T_1 = 0.024$ sec ; $T_2 = 0.2$ sec ; $T_3 = 2.0$ sec ; $K_{RA} = 8.8$; $K_{Mo} = 2.0$; $K_{TG} = 3.4$; $K_{RC} = 4.2$.

The following quantities are given: the conversion coefficient for the unit which converts analog quantities into unitary-code quantities $K_{c1} = 100$ units/mm; the conversion coefficient for the device which converts code quantities into a voltage (the decoder) $K_{c2} = 2$ v/units; the value of a pulse $\Delta = 0.01$ mm; the approximation error $\varepsilon_a = 0.005$ mm; and the input signal with respect to the Y coordinate y(t) = 1t.

In accordance with the graph for the transient response of the reduced limiting linear analog system (Fig. 7a),

$$\int_0^\infty |k(\tau)|\,d\tau = 1.68,$$

whence

$$|\varepsilon_{an}(\infty)_{max}|_{inv} \leqslant \Delta \int_0^\infty |k(\tau)|\,d\tau = 0.0168 \ \text{мм}.$$

As we know [2], the error $\varepsilon_d(\infty)$ is determined from the expression

$$\varepsilon_d(\infty)_{inv} = C_0 y(t) + C_1 \frac{dy(t)}{dt}. \tag{7}$$

In this case

$$C_0 = 0,\ C_1 = \frac{1}{NK_{c1}K_{c2}K_{RA}K_{Mo}} = \frac{1}{76} = 0.013 \text{ sec},$$
$$y(t) = 1t, \quad \frac{dy(t)}{dt} \equiv 1.$$

Substituting C_0, C_1, $y(t)$ and $dy(t)/dt$ into (7), we obtain $\varepsilon_d(\infty)_{inv} = 0.013$ mm.

Therefore

$$|\varepsilon_\Sigma(\infty)_{max}|_{inv} \leqslant |\varepsilon_d(\infty)_{inv}| + |\varepsilon_{an}(\infty)_{max}|_{inv} + |\varepsilon_a| = 0.0348 \ \text{мм}.$$

We see that the total error $|\varepsilon_\Sigma(\infty)_{max}|_{inv}$ is large. It consists chiefly of the errors $\varepsilon_{an}(\infty)_{max}$ and $\varepsilon_d(\infty)$. Therefore a reduction of these components is of great significance.

From Eq. (3) and (7) it is evident that a reduction of the oscillatory nature of the transient response leads directly to a reduction of the error $\varepsilon_{an}(t)_{max}$; an increase in the gain leads to a reduction of the error $\varepsilon_d(t)$.

Since the gain $K_p = K_{RA}K_{Mo}K_{TG}K_{RC}$ in a given system is sufficiently large, it can be assumed over a wide frequency range that

$$Lm\,|W_2(j\omega)| \approx 0, \quad \arg W_2(j\omega) \approx 0. \tag{8}$$

This is evident from the graph of the logarithmic frequency response for the equivalent open-loop analog system (Fig. 8).

Therefore in the intermediate frequency range which characterizes the transient response of the system the transfer function for the analog section can be written as

$$NW(s) = K_{an}W_1(s)\,W_2(s) \approx K_{an}W_1(s) = K_{an}\frac{1+T_3 s}{s^2}, \tag{9}$$

and the transfer function for the equivalent analog section of the system can be written as

$$K_{c1}K_{c2}NW(s) \approx K_{c1}K_{c2}K_{an}\frac{1+T_3 s}{s^2} = \beta\,\frac{1+T_3 s}{s_2}. \tag{10}$$

The characteristic equation for the equivalent closed-loop system will be

$$s^2 + \beta T_3 s + \beta = 0.$$

The roots of the latter equation are

$$\lambda_{1,2} = -\frac{\beta T_3}{2} \pm \sqrt{\frac{\beta^2 T_3^2}{4} - \beta} \approx -\beta \pm \sqrt{\beta^2 - \beta}. \tag{11}$$

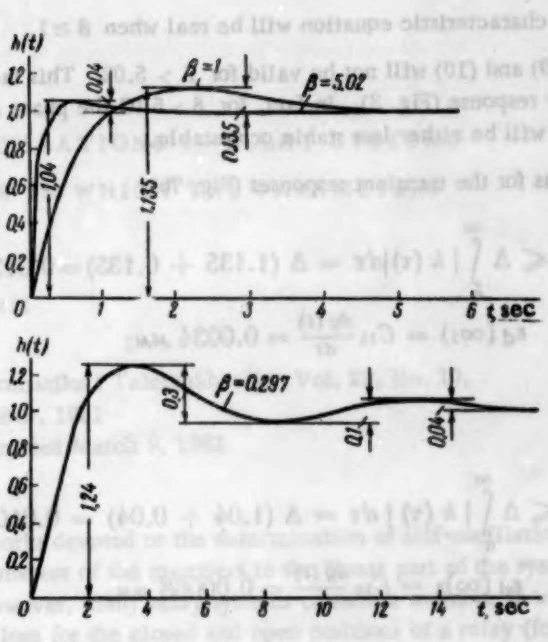


Fig. 7

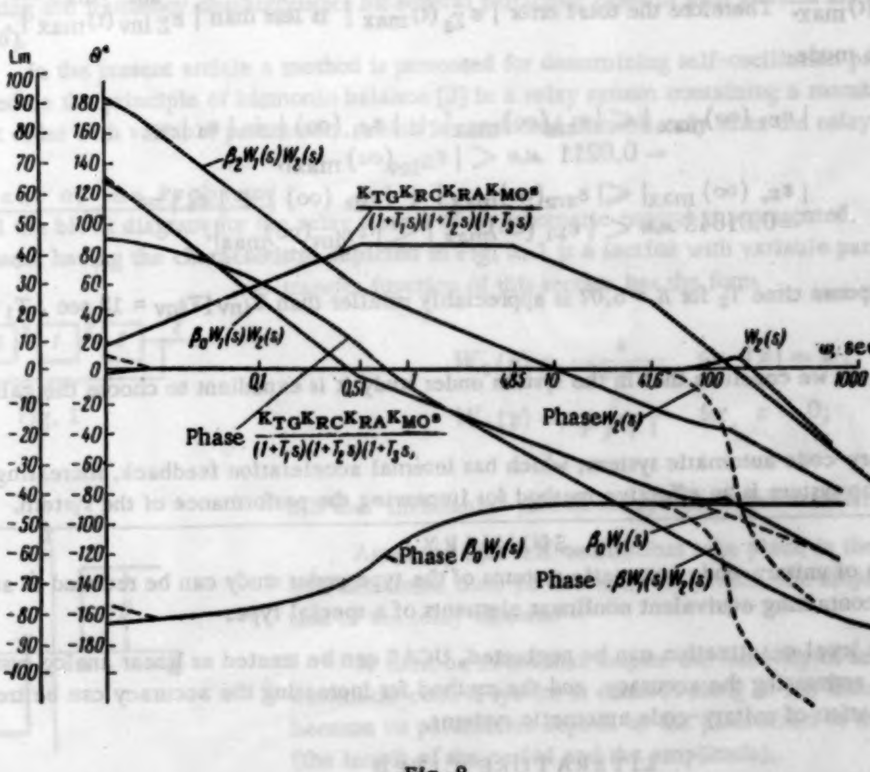


Fig. 8

In our case

$$\beta = K_{c1} K_{c2} K_{an} = 0.297 < 1,$$

and therefore the roots λ_1, λ_2 of the characteristic equation are complex.

Thus it follows from expression (11) that the coefficient β increases (this causes a more rapid attenuation of the oscillations) as the angular frequencies of the oscillations decrease. Therefore as gain β increases, the error $\varepsilon_{an}(t)_{max}$ decreases.

In that case the roots of the characteristic equation will be real when $\beta \geq 1$.

It should be noted that Eq. (9) and (10) will not be valid for $\beta > 5.02$. This fact can be established from the graph of the logarithmic frequency response (Fig. 8). In fact, for $\beta > 5.02$ the phase and amplitude margins become inadequate. Therefore the system will be either less stable or unstable.

In accordance with the graphs for the transient responses (Fig. 7b):

for $\beta = 1$

$$|\varepsilon_{an}(\infty)_{max}| \leqslant \Delta \int_0^\infty |k(\tau)| d\tau = \Delta (1.135 + 0.135) = 0.0127 \text{ мм},$$

$$\varepsilon_d(\infty_1) = C_{11} \frac{dy(t)}{dt} = 0.0034 \text{ мм};$$

for $\beta = 5.02$

$$|\varepsilon_{an}(\infty)_{max}|_2 \leqslant \Delta \int_0^\infty |k(\tau)| d\tau = \Delta (1.04 + 0.04) = 0.0108 \text{ мм},$$

$$\varepsilon_d(\infty)_2 = C_{12} \frac{dy(t)}{dt} = 0.00068 \text{ мм}.$$

The overall regulation for $\beta = 5.02$ is insignificant; here the errors $\varepsilon_d(t)_2$ and $\varepsilon_{an2}(t)_{max}$ are much less than $\varepsilon_d(t)_{inv}$ and $\varepsilon_{an.inv}(t)_{max}$. Therefore the total error $|\varepsilon_{\Sigma_2}(t)_{max}|$ is less than $|\varepsilon_{\Sigma inv}(t)_{max}|$.

In a steady-state mode:

$$|\varepsilon_{\Sigma 1}(\infty)_{max}| \leqslant |\varepsilon_{an1}(\infty)_{max}| + |\varepsilon_{d1}(\infty)| + |\varepsilon_a| = 0.0211 \text{ мм} < |\varepsilon_{\Sigma inv}(\infty)_{max}|,$$

$$|\varepsilon_{\Sigma_2}(\infty)_{max}| \leqslant |\varepsilon_{an2}(\infty)_{max}| + |\varepsilon_{d2}(\infty)| + |\varepsilon_a| = 0.01648 \text{ мм} < |\varepsilon_{\Sigma 1}(\infty)_{max}| < |\varepsilon_{\Sigma inv}(\infty)_{max}|.$$

The transient response time T_2 for $\beta = 5.02$ is appreciably smaller than T_{inv} [$T_{inv} = 13$ sec, $T_1 = 5$ sec, $T_2 = 1.2$ sec (Fig. 7)].

Based on the above, we conclude that in the system under study it is expedient to choose the gain within the limits $1 < \beta \leq 5.02$.

Thus in our unitary-code automatic system, which has internal acceleration feedback, increasing the gain of the equivalent open-loop system is an effective method for improving the performance of the system.

SUMMARY

The investigation of unitary-code automatic systems of the type under study can be reduced to an investigation of analog systems containing equivalent nonlinear elements of a special type.

In the case where level quantization can be neglected, UCAS can be treated as linear analog systems. The method cited above for estimating the accuracy, and the method for increasing the accuracy can be treated as a first step in the investigation of unitary-code automatic systems.

LITERATURE CITED

1. Ya. Z. Tsypkin, "Estimating the effect of level quantization on processes in digitial automatic systems," Avtomatika i Telemekhanika, 21, 3 (1960).
2. Fundamentals of Automatic Control Theory. Edited by V. V. Solodovnikov [in Russian] (Machinery Press, 1954).
3. B. V. Anisimov, Problems in the Theory and Design of Digital Systems for Program Control of Metal-Cutting Machine Tools. Dissertation for a Doctorate in Technical Sciences. Bauman Moscow Higher Technical School (1959).

ANALYSIS OF SELF-OSCILLATIONS IN RELAY SYSTEMS CONTAINING A MEMBER IN WHICH THE PARAMETERS CHANGE IN STEPS

V. I. Teverovskii

(Dnepropetrovsk)
Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1340-1344, October, 1961
Original article submitted March 9, 1961

In the majority of works devoted to the determination of self-oscillation parameters it is assumed that the parameters of the members in the linear part of the system stay constant with respect to time. However, many relay systems contain a member in which the parameters take on different values for the closed and open positions of a relay (for example, systems with relay servomechanisms, vibrating regulators of electrical machines). Methods of determining the frequency characteristics for several particular cases of such systems are given in [1, 6].

In the present article a method is presented for determining self-oscillation parameters based on the principle of harmonic balance [2] in a relay system containing a member of the first order with variable parameters, which is connected immediately after the relay element.

1. Statement of the Problem

In Fig. 1 the block diagram for the relay system of an automatic control is represented. In this illustration RE is a relay element having the characteristic depicted in Fig. 2; 1 is a section with variable parameters, and the transfer function of this section has the form

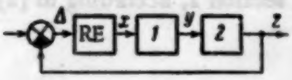


Fig. 1

$$W_1(p) = \frac{k}{Tp+1} \quad \text{for} \quad |x| = B,$$

$$W_1(p) = \frac{k'}{T'p+1} \quad \text{for} \quad x = 0;$$

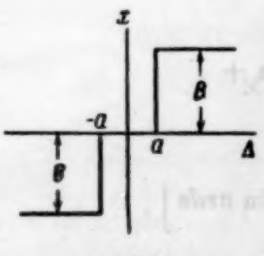


Fig. 2

2 is the "invariable" part of the system with a transfer function W(p).

Assuming that self-oscillations take place in the closed system, we will determine their parameters—the period, the amplitude, and the duration of the relay closure.

It must be noted that despite the linearity of section 1, in the closed automatic control system it cannot belong to the linear part of the system because its parameters depend on the parameters of the self-oscillations (the length of the period and the amplitude).

We will limit ourselves to a consideration of symmetrical self-oscillations, and for this case we will find the equivalent complex amplification coefficient of RE and of section 1.

2. Determination of the Output Signal of Section 1

The output signal of the relay element has the shape depicted in Fig. 3.

We will designate the period of the self-oscillations by T_0, the duration of the relay closure by γT_0. The output coordinate of section 1 is the steady reaction of it to a series of rectangular pulses with alternating signs.

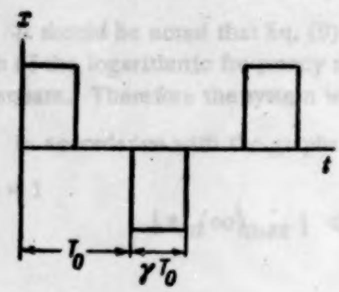


Fig. 3

To determine this reaction the results of pulse system theory can be used. A similar approach to the analysis of relay systems is used in [5]. In the present case section 1 has variable parameters, and so we will take advantage of the results in reference [3] for the determination of its output coordinates. First we will convert to dimensionless time $\bar{t} = t/T_0$. We will introduce also the designation $\varepsilon = \bar{t} - nT_0$, where $n = 0, 1, 2, 3, \ldots, 0 \leq \varepsilon \leq 1$.

On the basis of [3] the discrete transfer function of the pulse system formed by section 1 is written in the form

$$W_I^*(q, \varepsilon) = k\left(1 - \frac{e^q - e^{-\beta'(1-\gamma)}}{e^q - e^{-\beta_E}} e^{-\beta\varepsilon}\right) \quad (0 < \varepsilon < \gamma). \tag{2.1}$$

$$W_{II}^*(q, \varepsilon) = k\,\frac{(1 - e^{-\beta\gamma})\, e^{-\beta'(\varepsilon-\gamma)} e^q}{e^q - e^{-\beta_E}} \quad (\gamma < \varepsilon < 1),$$

where

$$\beta = \frac{T_0}{T}, \qquad \beta' = \frac{T_0}{T'}, \qquad \beta_E = \beta\gamma + \beta'(1 - \gamma).$$

The coordinate $\underline{x}$ can be represented as a multistage function

$$x(n) = B \cos \pi n. \tag{2.2}$$

Based on (2.1) and (2.2) we get for the output coordinate of section 1 for steady operation the expressions

$$y^I(n, \varepsilon) = Bk\left(1 - \frac{1 + e^{-\beta'(1-\gamma)}}{1 + e^{-\beta_E}} e^{-\beta\varepsilon}\right)(-1)^n \quad (0 < \varepsilon < \gamma). \tag{2.3}$$

$$y^{II}(n, \varepsilon) = Bk(-1)^n \frac{(1 - e^{-\beta\gamma})\, e^{-\beta'(\varepsilon-\gamma)}}{1 + e^{-\beta_E}} \quad (\gamma < \varepsilon < 1).$$

3. Complex Amplification Coefficient for the Non-Linear Part of the System

The equivalent complex amplification coefficient of the RE in common with section 1, according to [2], has the form

$$I(\omega, A) = g(A, \omega) + jb(A, \omega), \tag{3.1}$$

where $\omega = \pi/T_0$.

$$g(A, \omega) = \frac{2}{A}\left[\int_0^{\frac{1-\gamma}{2}} y^{II}(n, \varepsilon)\, e^{\frac{\beta'(1-\gamma)}{2}} \sin \pi\varepsilon\, d\varepsilon + \right.$$

$$\left. + \int_{\frac{1-\gamma}{2}}^{\frac{1+\gamma}{2}} y^I(n, \varepsilon)\, e^{\frac{\beta(1-\gamma)}{2}} \sin \pi\varepsilon\, d\varepsilon + \int_{\frac{1+\gamma}{2}}^{1} y^{II}(n, \varepsilon)\, e^{\frac{\beta'(1-\gamma)}{2}} \sin \pi\varepsilon\, d\varepsilon\right], \tag{3.2}$$

$$b(A, \omega) = \frac{2}{A}\left[\int_0^{\frac{1-\gamma}{2}} y^{II}(n, \varepsilon)\, e^{\frac{\beta'(1-\gamma)}{2}} \cos \pi\varepsilon\, d\varepsilon + \right.$$

$$\left. + \int_{\frac{1-\gamma}{2}}^{\frac{1+\gamma}{2}} y^I(n, \varepsilon)\, e^{\frac{\beta(1-\gamma)}{2}} \cos \pi\varepsilon\, d\varepsilon + \int_{\frac{1+\gamma}{2}}^{1} y^{II}(n, \varepsilon)\, e^{\frac{\beta'(1-\gamma)}{2}} \cos \pi\varepsilon\, d\varepsilon\right]. \tag{3.3}$$

Carrying out the integration in (3.2) and (3.3) we obtain

$$g = \frac{2B}{\pi A}\frac{k}{(\pi^2+\beta^2)(\pi^2+\beta'^2)}\left[\frac{(\pi^2-\beta\beta')(\beta'-\beta)(1-e^{-\beta\gamma})(1+e^{-\beta'(1-\gamma)})}{1+e^{-\beta_E}}\pi\cos\frac{\pi\gamma}{2}+\right.$$

$$\left.+(\pi^2\beta'^2+\pi^2\beta^2+2\beta'^2\beta^2)\sin\frac{\pi\gamma}{2}-\frac{(\beta'^2-\beta^2)\pi^2(e^{-\beta\gamma}+e^{-\beta'(1-\gamma)})}{1+e^{-\beta_E}}\sin\frac{\pi\gamma}{2}\right], \tag{3.4}$$

$$b = \frac{2B}{A\pi}\frac{k}{(\pi^2+\beta^2)(\pi^2+\beta'^2)}\left[(\pi^2-\beta\beta')(\beta'-\beta)\frac{e^{-\beta\gamma}+e^{-\beta'(1-\gamma)}}{1+e^{-\beta_E}}\pi\sin\frac{\pi\gamma}{2}-\right.$$

$$\left.-\pi(\pi^2+\beta\beta')(\beta+\beta')\sin\frac{\pi\gamma}{2}+\frac{(1-e^{-\beta\gamma})(1+e^{-\beta'(1-\gamma)})(\beta'^2-\beta^2)}{1+e^{-\beta_E}}\pi^2\cos\frac{\pi\gamma}{2}\right]. \tag{3.5}$$

Taking into account that when the signal $\Delta = A\sin\frac{\pi}{T_0}t$ is acting on the input of the RE we have the relation

$$\gamma = \frac{2}{\pi}\arccos\frac{a}{A},$$

and also that $\beta = \frac{\pi}{\omega T}$, $\beta' = \frac{\pi}{\omega T'}$, we obtain

$$g = \frac{2B}{\pi A}\frac{k}{(1+\omega^2T^2)(1+\omega^2T'^2)}\left[\frac{\omega(\omega^2TT'-1)(T-T')\left(1-e^{-\frac{\pi\gamma}{\omega T}}\right)\left(1+e^{-\frac{\pi(1-\gamma)}{\omega T'}}\right)}{1+e^{-\frac{\pi}{\omega T_E}}}\frac{a}{A}+\right.$$

$$+(T^2\omega^2+T'^2\omega^2+2)\sqrt{1-\frac{a^2}{A^2}}-$$

$$\left.-\omega^2(T^2-T'^2)\sqrt{1-\frac{a^2}{A^2}}\frac{\left(e^{-\frac{\pi\gamma}{\omega T}}+e^{-\frac{\pi(1-\gamma)}{\omega T'}}\right)}{1+e^{-\frac{\pi}{\omega T_E}}}\right]; \tag{3.6}$$

$$b = \frac{2B}{\pi A}\frac{k}{(1+\omega^2T^2)(1+\omega^2T'^2)}\left[\frac{\omega(\omega^2TT'-1)(T-T')\left(e^{-\frac{\pi\gamma}{\omega T}}+e^{-\frac{\pi(1-\gamma)}{\omega T'}}\right)}{1+e^{-\frac{\pi}{\omega T_E}}}\times\right.$$

$$\times\sqrt{1-\frac{a^2}{A^2}}-\omega(1+TT'\omega^2)(T+T')\sqrt{1-\frac{a^2}{A^2}}+$$

$$\left.+\frac{\left(1-e^{-\frac{\pi\gamma}{\omega T}}\right)\left(1+e^{-\frac{\pi(1-\gamma)}{\omega T'}}\right)(T^2-T'^2)\omega^2}{1+e^{-\frac{\pi}{\omega T_E}}}\frac{a}{A}\right], \tag{3.7}$$

where $T_E = \frac{TT'}{T'\gamma+T(1-\gamma)}$.

We will consider a series of special cases of these expressions:

a) $\omega \to 0, \quad g \approx \frac{4B}{\pi A}\sqrt{1-\frac{a^2}{A^2}}, \quad b \approx 0.$

Moreover, it is easily seen that $I(\omega, A)$ is the equivalent complex amplification coefficient of the RE.

b) $\omega \to \infty$, $g \to 0$, $b \to 0$, consequently, $I(\omega, A) \to 0$.

c) $$\frac{a}{A} \to 1,\ \gamma \to 0,\ I(\omega, A) \to 0.$$

d) $$\frac{a}{A} \to 0,\ \gamma \approx 1,\quad g \approx \frac{4Bk\sqrt{1-\frac{a^2}{A^2}}}{\pi A(1+\omega^2T^2)} \to 0,$$

$$b = -\frac{4Bk}{\pi A}\sqrt{1-\frac{a^2}{A^2}}\,\frac{\omega T}{\omega^2T^2+1} \to 0.$$

e) $$T = T',\ g = \frac{4Bk}{\pi A}\,\frac{1}{\omega^2T^2+1}\sqrt{1-\frac{a^2}{A^2}},\ b = -\frac{4Bk}{\pi A}\,\frac{\omega T}{\omega^2T^2+1}\sqrt{1-\frac{a^2}{A^2}}.$$

This case corresponds to a series connection of the RE and the section with constant parameters (k'= k, T' = T).

f) $$T' \to \infty,\ g = \frac{4Bk}{\pi A}\,\frac{\sqrt{1-\frac{a^2}{A^2}} - \frac{a}{A}\,\omega T\,\mathrm{th}\left(\arccos\frac{a}{A}\Big/\omega T\right)}{1+\omega^2T^2},$$

$$b = -\frac{4Bk}{\pi A}\,\frac{\omega T\sqrt{1-\frac{a^2}{A^2}} + \frac{a}{A}\,\mathrm{th}\left(\arccos\frac{a}{A}\Big/\omega T\right)}{1+\omega^2T^2}.$$

This result concurs with the result in reference [1].

Expressions (3.6) and (3.7) obtained for the equivalent complex amplification coefficient permit the problem of determining the self-oscillation parameters to be solved by the method of harmonic balance.

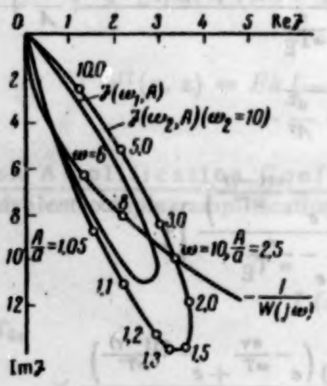


Fig. 4

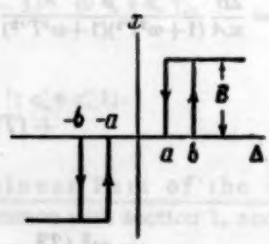


Fig. 5

The starting expression for this calculation is the well-known relation [2]

$$I(\omega, A) = \frac{-1}{W(j\omega)}. \tag{3.8}$$

A solution of equation (3.8) is produced graphically by the following means.

In the complex plane (g, jb) a family of root loci $I(\omega, A)$ is plotted, each of which corresponds to a fixed value of ω (Fig. 4).* The construction is carried out by means of calculations of the components of $I(\omega, A)$ according to formulas (3.6) and (3.7) when A is varied from a to $+\infty$ for every value of ω = const. A root locus of the function $-1/W(j\omega)$ is constructed on these same coordinates. The self-oscillation parameters are the values of A and ω corresponding to the point of intersection of the root locus $-1/W(j\omega)$ with that one of the root loci $I(\omega, A)$ for which the value of ω corresponds to the value of ω of the function $-1/W(j\omega)$ at this point.

The stability of the self-oscillations with the parameters determined in this way is verified by the same method as in the case of a non-linear system with constant parameters [2].

If the characteristic of the relay element has the shape shown in Fig. 5, the corresponding root locus $I'(\omega, A)$ can be obtained from the root locus $I(\omega, A)$ by rotating all vectors through the angle

* Note that from cases c) and d) it follows that the root loci $I(\omega, A)$ are closed for all values of ω.

$$\Delta\psi = (\alpha_1 - \alpha_2) / 2\omega,$$

where $\alpha_1 = \arcsin(a/A)$, $\alpha_2 = \arcsin(b/A)$ and by substituting $a_E = A \sin(\alpha_1 + \alpha_2) / 2$ (see [1]).

Thus the problem of finding self-oscillation parameters in a relay system containing a member with stepwise changing parameters comes to the construction of a family of root loci $I(\omega, A)$ and a root locus of the inverse amplitude-phase characteristic of the constant part of the system.

LITERATURE CITED

1. V. A. Bessekerskii, "On the transfer function of a relay actuating mechanism" [in Russian] Izv. AN SSSR, OTN, Énergetika i Avtomatika, No. 1 (1959).
2. L. S. Gol'dfarb, "A method of analyzing non-linear control systems based on the principle of harmonic balance Coll.:"Principles of Automatic Control" [in Russian] Theory. Edited by V. V. Solodovnikov, Mashgiz (1954).
3. V. I. Teverovskii, "On a special case of a pulse system with variable parameters which change in steps" [in Russian] Avtomatika i Telemekhanika, 21, No. 1 (1960).
4. Ya. Z. Tsypkin, The Theory of Pulse Systems [in Russian] (Fizmatgiz, 1958).
5. Ya. Z. Tsypkin, Transient and Steady Processes in Pulse Circuits [in Russian] (Gosénergoizdat, 1951).
6. V. A. Bessekerskii and V. B. Diomidov, "Transfer function of a relay actuating mechanism with two-phase asynchronous motor"[in Russian] (Scientific-Technical Information for Apparatus Construction. No. 4 (8), Sudpromgiz, 1958).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

ON THE BEHAVIOR OF FINITE AUTOMATA IN RANDOM MEDIA

M. L. Tsetlin

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1345-1354, October, 1961
Original article submitted April 1, 1961

The paper studies the functioning of finite automata in media that respond to their actions in a random fashion. The concept of the expediency of automaton behavior is introduced and examples are cited for automata which have expedient behavior.

The behavior of automata in media whose probability properties depend on time in accordance with a Markov chain is studied.*

Introduction

In this paper we study the functioning (behavior) of finite automata in media that respond to their actions in a random fashion.

It is assumed that an automaton perceives any response of the medium as a reproof ("penalty") for its action or as the absence of such a reproof (a "non-penalty"); it is also assumed that the medium "penalizes" the automaton with a probability that is specified for each of its actions. For these assumptions it will be demonstrated that the behavior of a finite automaton in a stationary random medium is described by a finite Markov chain; the magnitude of the mathematical expectation of the penalty serves as the measure of expediency for the behavior of the automaton.

The paper cites examples of automata that have expedient behavior. In particular, an automaton construction (an automaton with a linear tactic) is indicated for which the mathematical expectation of a penalty decreases as the memory capacity increases and coincides in the limit with the minimal possible expectation in the specified random medium. Such an automaton performs (with a probability close to unity) that action for which the probability of a penalty is minimal in the specified medium.

Then we study the behavior of finite automata in a random medium whose probability properties depend on time in accordance with a Markov chain. We investigate the behavior of an automaton with a linear tactic in this random medium. It is demonstrated in particular that the mathematical expectation of a penalty in this case reaches the minimum for a certain fixed memory capacity and increases as it is increased further. The existence of an optimal memory capacity in a given medium for automata with a linear tactic makes it possible to design automata which have the most expedient behavior in the specified random medium.

The paper cites graphs that indicate the dependence of the mathematical expectation of a penalty on the memory capacity and parameters of the medium; we also provide a table which specifies the optimal values of memory capacity and the corresponding values of the mathematical expectation of a penalty for various random media.

1. The Behavior of Automata in Stationary Random Media

Assume that the finite automaton A is described by its canonical equations**

$$\varphi(t+1) = \Phi(\varphi(t), s(t+1)), \qquad (1)$$

* A brief presentation of the results is given in [1].

** The definition of a finite automaton used here coincides with the definition of a non-primitive network cited in [2, 3]. It is not difficult to arrive at the same definition if we start from the definition of a restrictedly-deterministic operator in accordance with the analysis performed by N. E. Kobrinskii and B. A. Trakhtenbrot [4, 5] and then study the set of restrictively-deterministic operators which can be distinguished solely by their initial states.

$$f(t) = F(\varphi(t)) \qquad (t = 1, 2, \ldots). \tag{2}$$

We shall assume that the input variable can acquire only two values: 0 and 1. The value $s = 1$ shall be called a "penalty," and the value $s = 0$ shall be called "non-penalty."

We shall assume further that the output variable $f(t)$ of the automaton can acquire $\underline{k}$ different values $f_1, \ldots, f_k$. The values of the variable $f(t)$ shall be called the "actions" of the automaton, and we shall say that at the instant $\underline{t}$ the automaton has performed the $\underline{\alpha}$th action if $f(t) = f_\alpha$ ($\alpha = 1, \ldots, k$).

The values of the variable $\varphi(t)$ are called the "states" of the automaton. We shall assume that the automaton A can have $\underline{m}$ states $\varphi_1, \ldots, \varphi_m$ ($m \geq k$), and we shall say that the automaton is in the $\underline{j}$th state at the instant $\underline{t}$ if $\varphi(t) = \varphi_j$. The number $\underline{m}$ shall be called the "memory capacity" of the automaton. The action f_α is called an action corresponding to the state φ_j if $f_\alpha = F(\varphi_j)$.

Equation (2) describes the dependence of the automaton actions on its states under these conditions; Eq. (1) describes the transitions of the automaton states due to the effect of the input variables. Since the input variable $s(t)$ can acquire two values, Eq. (1) describes a pair of representations of the set of states for the finite automaton A in itself; one of these representations is specified for $s = 1$, and the other is specified for $s = 0$.

For purposes of our subsequent analysis it will be convenient to describe these representations using a special matrix of states for the automaton: $A(s) = \|a_{ij}(s)\|$ ($i, j = 1, \ldots, m$).

Each row in this matrix contains exactly one element equal to unity for any fixed value of $\underline{s}$; the remaining elements of this row are zeros. Such matrices are called simple matrices in [2, 3]. The matrix of states $A(s)$ for the finite automaton A determines the transitions of states in the following manner: if at the instant $\underline{t}$ the automaton was in the state φ_i, then at the instant $t + 1$ the automaton will make the transition to a state φ_l for which $a_{il}[s(t + 1)] = 1$.

From the matrix $A(s)$ it is possible to restore Eq. (1) single-valuedly in such a way that the matrix of states and Eq. (2) fully define the finite automaton A.*

The transitions of states for each value of the input variable $s(t)$ can be conveniently depicted by graphs of state.** These graphs are related to the matrix $A(s)$ in the following manner. Each state φ_i of the automaton A is juxtaposed with a vertex $\underline{i}$ of the graph, and each non-zero matrix element $a_{ij}(s)$ is juxtaposed with an arrow directed from the vertex $\underline{i}$ to the vertex $\underline{j}$. From the simplicity of the matrix $A(s)$ it follows under these conditions that exactly one arrow emanates from each vertex of the graph. For automata which are studied in this paper the transitions of states are described by a pair of such graphs (for $s = 1$ and for $s = 0$). From the graphs of states it is possible to restore the matrix $A(s)$ in a single-valued manner.

We shall now proceed to study the functioning of automata in media that respond in random fashion to their actions.

We shall state that the automaton A functions in the stationary random medium $C = C(p_1, \ldots, p_k)$, if the values of the input variable and the actions of the automaton are related as follows: the action f_α ($\alpha = 1, \ldots, k$) performed by the automaton at the instant $\underline{t}$ generates the value $s = 1$ (a penalty) at the instant $t + 1$ with the probability p_α, and the value $s = 0$ (a non-penalty) with the probability $q_\alpha = 1 - p_\alpha$.

Assume that at the instant $\underline{t}$ the automaton was in the state φ_i ($i = 1, \ldots, m$), which corresponds to the action $f_{\alpha_i} = F(\varphi_i)$ ($\alpha_i = 1, \ldots, k$). Then the probability p_{ij} that the automaton will make the transition from the state φ_i to the state φ_j is determined from the formula

$$p_{ij} = p_{\alpha_i} a_{ij}(1) + q_{\alpha_i} a_{ij}(0) \qquad (i, j = 1, \ldots, m). \tag{3}$$

The matrix $P = \|p_{ij}\|$ of the transition probabilities is stochastic; this can easily be demonstrated by making use of the simplicity of the matrix $A(s)$. Thus the functioning of the automaton in a stationary random medium is described by a finite Markov chain. Without any substantial limitation of the general nature of our analysis we can

* For more details cf. [2, 3].

** Graphs of this type were used in the paper by V. I. Shestakov [6].

assume that this chain is ergodic [i.e., we can assume that there exist final (limiting) probabilities for the states of the automaton in the specified medium which do not depend on its initial state].

We shall use r_i to denote the final probability of the state φ_i ($i = 1, \ldots, m$), and σ_α ($\alpha = 1, \ldots, k$) shall denote the sum of the final probabilities for such states φ_i which correspond to the action f_α [i.e., states for which $F(\varphi_i) = f_\alpha$].

Then the mathematical expectation $M = M(A, C)$ of a penalty for the automaton A in the medium C is expressed by the formula

$$M(A, C) = \sum_{\alpha=1}^{k} p_\alpha \sigma_\alpha. \tag{4}$$

Henceforth we shall make use of the substitutions

$$M_{\max} = \max(p_1, \ldots, p_k), \quad M_{\min} = \min(p_1, \ldots, p_k), \quad M_0 = \frac{p_1 + \cdots + p_k}{k}.$$

Then it is obvious that

$$M_{\min} \leqslant M \leqslant M_{\max}. \tag{5}$$

The expediency of the automaton behavior resides in the lowering of the quantity M; it can be defined as the closeness of the quantity M to $M_{\min}$. An automaton for which $M = M_0$ is naturally called an automaton which does not have expedient behavior. When an automaton with expedient behavior functions in any stationary medium the average penalty M meted out to it will be less than M_0 (excluding the case where the probabilities of penalties for all actions are equal).

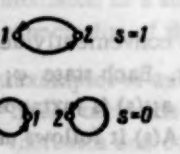


Fig. 1

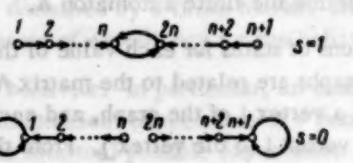


Fig. 2

Example 1. The automaton $L_{2,2}$ has two states φ_1 and φ_2 and two actions f_1 and f_2; $F(\varphi_1) = f_1$, $F(\varphi_2) = f_2$. The matrix of states for this automaton will be

$$A(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., the states of the automaton will change for a penalty and will remain unchanged for a non-penalty; the graphs of states for this automaton are shown in Fig. 1. We shall study the functioning of this automaton in the medium $C = C(p_1, p_2)$. Having formulated the matrix of transitional probabilities in accordance with (3), we obtain

$$P = \begin{pmatrix} q_1 & p_1 \\ p_2 & q_2 \end{pmatrix}.$$

From this we find the following equations for computing the final probabilities r_1 and r_2:

$$r_1 = q_1 r_1 + p_2 r_2, \quad r_2 = p_1 r_1 + q_2 r_2.$$

The normalization condition is given by

$$r_1 + r_2 = 1.$$

Having computed the values $r_1 = p_2/(p_1 + p_2)$, $r_2 = p_1/(p_1 + p_2)$, we find the mathematical expectation of a penalty in accordance with (4):

$$M(L_{2,2}, C) = p_1 r_1 + p_2 r_2 = \frac{2p_1 p_2}{p_1 + p_2}.$$

It is not difficult to verify the fact that $M(L_{2,2}, C) \leq M_0$ (equality occurs only for $p_1 = p_2$).

Thus the automaton $L_{2,2}$ has expedient behavior.

Example 2. The automaton $L_{2n,2}$ (an automaton with a linear tactic) is a natural generalization of the automaton $L_{2,2}$; it has 2n states $\varphi_1, \ldots, \varphi_{2n}$ and two different actions f_1, f_2. Under these conditions

$$F(\varphi_1) = F(\varphi_2) = \ldots = F(\varphi_n) = f_1, \quad F(\varphi_{n+1}) = F(\varphi_{n+2}) = \ldots = F(\varphi_{2n}) = f_2. \tag{6}$$

The graphs of states for the automaton $L_{2n,2}$ are shown in Fig. 2; the matrix of states for this automaton has the elements $a_{ij}(s)$ which are determined according to the following formulas:

$$\begin{gathered} a_{ij}(0) = 1 \ \text{ for } i = 2, 3, \ldots, n, n+2, \ldots, 2n, \ j = i - 1; \ a_{11}(0) = a_{n+1\, n+1}(0) = 1; \\ a_{ij}(1) = 1 \ \text{ for } \ i = 1, 2, \ldots, n-1, \ n+1, \ldots, 2n-1 \text{ and } j = i + 1; \\ a_{n,2n}(1) = a_{2n,n}(1) = 1. \end{gathered} \tag{7}$$

The remaining elements are zeros.

We shall compute the mathematical expectation $M(L_{2n,2}, C)$ of a penalty for the automaton $L_{2n,2}$ which functions in the medium $C = C(p_1, p_2)$.

Having determined the transition probabilities from (3) and (7), we obtain the following system of equations for computing the final probabilities $r_j (j = 1, \ldots, 2n)$:

$$\begin{aligned} r_1 &= q_1 r_1 + q_1 r_2, & r_{n+1} &= q_2 r_{n+1} + q_2 r_{n+2}, \\ r_2 &= p_1 r_1 + q_1 r_3, & r_{n+2} &= p_2 r_{n+1} + q_2 r_{n+3}, \\ &\ldots\ldots\ldots & &\ldots\ldots\ldots \\ r_k &= p_1 r_{k-1} + q_1 r_{k+1}, & r_{n+k} &= p_2 r_{n+k-1} + q_2 r_{n+k+1}, \\ &\ldots\ldots\ldots & &\ldots\ldots\ldots \\ r_n &= p_1 r_{n-1} + p_2 r_{2n}, & r_{2n} &= p_2 r_{2n-1} + p_1 r_n \end{aligned} \tag{8}$$

and the normalization condition is given by

$$r_1 + r_2 + \ldots + r_{2n} = 1.$$

Writing the solution in the form $r_k = a_1 \lambda_1^{k-1}$, $r_{n+k} = a_2 \lambda_2^{k-1}$ ($k = 1, \ldots, n$), we obtain the following characteristic equations for determining the eigennumbers λ_1 and λ_2:

$$q_\alpha \lambda_\alpha^2 - \lambda_\alpha + p_\alpha = 0 \quad (\alpha = 1, 2).$$

Determining $\lambda_1^{(1)} = \lambda_2^{(1)} = 1$, $\lambda_1^{(2)} = p_1/q_1 = \alpha_1$, $\lambda_2^{(2)} = p_2/q_2 = \alpha_2$ from these equations, we shall seek solutions in the form

$$r_k = A_1 \alpha_1^{k-1} + B_1, \quad r_{n+k} = A_2 \alpha_2^{k-1} + B_2.$$

Making use of the equations for r_1 and r_{n+1}, we prove that $B_1 = B_2 = 0$; from the equations for r_n, r_{2n} it follows that

$$A_1 \frac{\alpha_1^n}{1 + \alpha_1} = A_2 \frac{\alpha_2^n}{1 + \alpha_2}. \tag{9}$$

We now compute the sums

$$\sigma_1 = \sum_{k=1}^{n} r_k = A_1 \sum_{k=1}^{n} \alpha_1^{k-1} = A_1 \frac{\alpha_1^n - 1}{\alpha_1 - 1},$$

$$\sigma_2 = \sum_{k=1}^{n} r_{n+k} = A_2 \sum_{k=1}^{n} \alpha_2^{k-1} = A_2 \frac{\alpha_2^n - 1}{\alpha_2 - 1}.$$

From the normalization conditions $\sigma_1 + \sigma_2 = 1$ and Eq. (9) we find the values of the coefficients A_1, A_2 and compute the mathematical expectation $M(L_{2n,2}, C)$ of a penalty on the basis of formula (4):

$$M(L_{2n,2}, C) = \frac{\dfrac{1}{p_1^{n-1}} \dfrac{p_1^n - q_1^n}{p_1 - q_1} + \dfrac{1}{p_2^{n-1}} \dfrac{p_2^n - q_2^n}{p_2 - q_2}}{\dfrac{1}{p_1^n} \dfrac{p_1^n - q_1^n}{p_1 - q_1} + \dfrac{1}{p_2^n} \dfrac{p_2^n - q_2^n}{p_2 - q_2}}. \tag{10}$$

It is important to note that $M(L_{2n,2}, C)$ is a decreasing function of the memory capacity and that for $M_{\min} \leq$ $\leq 1/2$

$$\lim_{n\to\infty} M(L_{2n,2}, C) = M_{\min} \tag{11}$$

This relationship means that for a sufficiently large memory capacity $\underline{n}$ the automaton $L_{2n,2}$ performs almost exclusively that action for which the probability of a penalty is minimal. Automata which have this property shall be called asymptotically optimal.

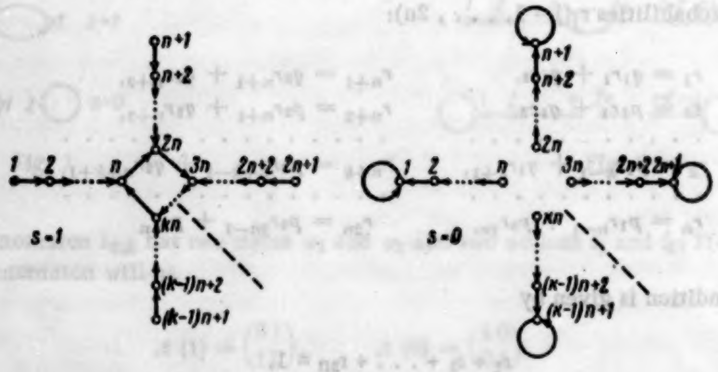


Fig. 3

Example 3. We shall study the automaton $L_{kn,k}$, which has kn states $\varphi_1, \ldots, \varphi_{kn}$ and $\underline{k}$ actions $f_1, \ldots, f_k$. Assume

$$F(\varphi_1) = F(\varphi_2) = \ldots = F(\varphi_n) = f_1,$$
$$F(\varphi_{n+1}) = F(\varphi_{n+2}) = \ldots F(\varphi_{2n}) = f_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F(\varphi_{(k-1)n+1}) = F(\varphi_{(k-1)n+2}) = \ldots = F(\varphi_{kn}) = f_k.$$

The graphs of states for this automaton are cited in Fig. 3. It is not difficult to note that for k = 2 the automaton $L_{kn,k}$ becomes the automaton with a linear tactic described in example 2.

In the medium $C = C(p_1, \ldots, p_k)$ the mathematical expectation $M(L_{kn,k}, C)$ of a penalty for the automaton $L_{kn,k}$ is expressed by the formula

$$
M\,(L_{kn,k},\ C) = \frac{\sum\limits_{\alpha=1}^{k} \dfrac{p_\alpha^n - q_\alpha^n}{p_\alpha - q_\alpha}\,\dfrac{1}{p_\alpha^{n-1}}}{\sum\limits_{\alpha=1}^{k} \dfrac{p_\alpha^n - q_\alpha^n}{p_\alpha - q_\alpha}\,\dfrac{1}{p_\alpha^{n}}}. \tag{12}
$$

The automaton $L_{kn,k}$ is also asymptotically optimal.

Example 4. We shall now describe the automaton $G_{2n,2}$ which has 2n states and two actions f_1 and f_2:

$$
F(\varphi_1) = F(\varphi_2) = \ldots = F(\varphi_n) = f_1,
$$
$$
F(\varphi_{n+1}) = F(\varphi_{n+2}) = \ldots = F(\varphi_{2n}) = f_2.
$$

The graphs for this automaton are shown in Fig. 4. Such an automaton is naturally called an automaton with a hysteresis tactic.

When this automaton functions in a medium $C(p,\ 1-p)$ the mathematical expectation of a penalty $M(G_{2n,2},\ C)$ is determined from a formula

$$
M\,(G_{2n,2},\ C) = \frac{\alpha}{\alpha+1}\,\frac{(\alpha^n-1)^2 + n\alpha^{n-1}(\alpha-1)^2}{(\alpha^n-1)(\alpha^{n+1}-1)} \quad \left(\alpha = \frac{p}{1+p}\right).
$$

It can be demonstrated that the automation $G_{2n,2}$ is asymptotically minimal but that

$$
M\,(\mathfrak{G}_{2n,2},\ C) \geqslant M\,(L_{2n,2},\ C),
$$

so that an automaton with a hysteresis tactic has lower expediency of behavior than an automaton with a linear tactic.

2. The Behavior of Automata in Composite Media

In the preceding section we studied the behavior of automata in media whose probability properties are independent of time. The expediency of the automaton behavior in such media is manifested in a lowering of the mathematical expectation of the penalty in those cases when the probability properties of the medium are not known in advance. The a priori knowledge of the constant $p_1, \ldots, p_k$ makes it senseless to design an automaton which is capable of performing a number of different actions depending on its state. In fact, knowing the values $p_1, \ldots, p_k$, it is possible to design an automaton which has just one state and performs one action — precisely that action for which the probability of a penalty is minimal.

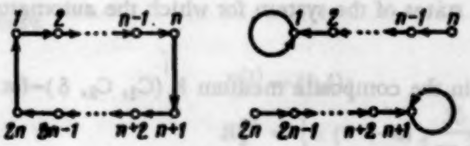


Fig. 4

In this section we shall study functioning of finite automata in a medium whose probability properties vary in random fashion. Then even if we know the possible probability characteristics of the medium we cannot design an automaton with expedient behavior that would have just one action.

We shall assume that the time dependence of the probability properties of the medium is specified in a special manner: it is assumed that the medium in which the automaton functions is composed of stationary random media whose switching is achieved by a Markov chain.

Thus we shall study the simplest composite medium $K = K(C_1,\ C_2,\ \delta)$ specified by a Markov chain with two states C_1 and C_2 and a transition probability matrix

$$
\Delta = \begin{pmatrix} 1-\delta & \delta \\ \delta & 1-\delta \end{pmatrix}.
$$

The state C_1 corresponds to the stationary random medium $C_1 = C(p_1^{(1)}, \ldots, p_k^{(1)})$; the state C_2 corresponds to the medium $C_2 = C(p_1^{(2)}, \ldots, p_k^{(2)})$.

We shall state that the automaton A functions in a composite medium K if at each instant it functions either in the medium C_1 or in the medium C_2 (i.e., if at each instant its actions and the values of the input variable are related in the fashion described above for one of the media). Under these conditions if the automaton was functioning in the medium C_β ($\beta = 1, 2$) at the instant $\underline{t}$, it follows that at the instant $t + 1$ it will function in the same medium with the probability $1-\delta$, and in the other medium with the probability δ.*

It is not difficult to prove that the parameter δ is the average frequency with which the states of the composite medium are switched. In fact, the average number T of cycles during which the state of the medium remains constant can be computed as follows:

$$T = \sum_{m=1}^{\infty} m\,(1-\delta)^{m-1}\,\delta = \frac{1}{\delta}. \tag{13}$$

We shall say that the system S consisting of the automaton and the composite medium is in the state $\psi_i^{(\beta)}$ ($i = 1, \ldots, m$; $\beta = 1, 2$) at the instant $\underline{t}$ if at that instant the automaton is in the state φ_i and the composite medium is in the state C_β. The probabilities for transitions between states of the system are described by the cell matrix $\Pi = \|\pi_{ij}\|$ ($i, j = 1, \ldots, m$):

$$\pi_{ij} = \begin{pmatrix} [P_{\alpha_i}^{(1)} a_{ij}(1) + q_{\alpha_i}^{(1)} a_{ij}(0)]\,(1-\delta) & [P_{\alpha_i}^{(1)} a_{ij}(1) + q_{\alpha_i}^{(1)} a_{ij}(0)]\,\delta \\ [p_{\alpha_i}^{(2)} a_{ij}(1) + q_{\alpha_i}^{(2)} a_{ij}(0)]\,\delta & [p_{\alpha_i}^{(2)} a_{ij}(1) + q_{\alpha_i}^{(2)} a_{ij}(0)]\,(1-\delta) \end{pmatrix}. \tag{14}$$

Here $p_{\alpha_i}^{(\beta)}$ is the probability of a penalty in the medium C_β for the action $f_{\alpha_i} = F(\varphi_i)$. Assume that the system under study is ergodic. Then the final probabilities $r_i^{(\beta)}$ of the states $\psi_i^{(\beta)}$ for the system S will not depend on its initial states. The mathematical expectation $M = M(A, K)$ of a penalty for the automaton A in the medium K can then be given by the formula

$$M(A, K) = \sum_{\alpha=1}^{k} (p_\alpha^{(1)}\sigma_\alpha^{(1)} + p_\alpha^{(2)}\sigma_\alpha^{(2)}), \tag{15}$$

where $\sigma_\alpha^{(\beta)}$ ($\alpha = 1, \ldots, k$; $\beta = 1, 2$) is the total probability of those states of the system for which the automaton performs the action f_α and the composite medium is in the state C_β.

We shall study the behavior of an automaton with a linear tactic in the composite medium $K(C_1, C_2, \delta)$—for example, the automaton $L_{2n,2}$ in example 2 of the preceding section.

We shall recall that this automaton has 2n states $\varphi_1, \ldots, \varphi_{2n}$ and two actions f_1, f_2. It is specified by the formulas (6) and (7), and its graph of states is shown in Fig. 2.

We shall compute the mathematical expectation of a penalty for the automaton $L_{2n,2}$ under conditions where we have assumed for simplicity that

$$C_1 = C(p, 1-p), \quad C_2 = C(1-p, p). \tag{16}$$

In order to determine the final probabilities of the states for the system we have a set of equations analogous to (8); however, in this system the probabilities r_i ($i = 1, \ldots, 2n$) of the states of the automaton $L_{2n,2}$ in a stationary medium are replaced by the vectors $R_i = (r_i^{(1)}, r_i^{(2)})$ ($i = 1, \ldots, 2n$) for the probabilities of the system states; the quantities p_1, p_2, q_1, q_2 are replaced by the corresponding matrices.

* Of course, it is possible to study the composite medium $K = K(C_1, \ldots, C_s, D)$ that has $\underline{s}$ states $C_1, \ldots, C_s$ corresponding to different stationary random media, and the transition probability matrix D. However, in this paper we limit ourselves to the simplest case where the matrix D is symmetrical and $s = 2$.

The set of equations is written in the following manner when condition (16) is taken into account:

$$\begin{aligned} &R_1 = QR_1 + QR_2, && R_{n+1} = SR_{n+1} + SR_{n+2}, \\ &R_2 = SR_1 + QR_3, && R_{n+2} = QR_{n+1} + SR_{n+3}, \\ &\ldots\ldots\ldots && \ldots\ldots\ldots \\ &R_k = SR_{k-1} + QR_{k+1}, && R_{n+k} = QR_{n+k-1} + SR_{n+k+1}, \\ &\ldots\ldots\ldots && \ldots\ldots\ldots \\ &R_n = SR_{n-1} + QR_{2n}, && R_{2n} = QR_{2n-1} + SR_n, \end{aligned} \tag{17}$$

where

$$Q = \begin{pmatrix} (1-\delta)\,q & \delta p \\ \delta q & (1-\delta)\,p \end{pmatrix}, \quad S = \begin{pmatrix} (1-\delta)\,p & \delta q \\ \delta p & (1-\delta)\,q \end{pmatrix} \quad (q = 1 - p). \tag{18}$$

We shall seek the solution in the form

$$R_k = R_0\lambda^{k-1}, \quad R_{n+k} = R_0\lambda^{2n-k} \quad (k = 1, \ldots, n),$$

where $R_0 = r_0^{(1)}, r_0^{(2)}$ is a constant vector.

In order to determine the eigennumbers λ and the eigenvectors we make use of the characteristic equation

$$(Q\lambda^2 - \lambda E + S)\,R_0 = 0, \tag{19}$$

where E is a unit matrix of the second order.

From the equation $\mathrm{Det}\,(Q\lambda^2 - \lambda E + S) = 0$ we find four values of the eigennumbers: $\lambda_1 = \lambda_2 = 1$, and λ_3 and λ_4 which are roots of the quadratic equation

$$x^2 - 2x\left(\frac{(\alpha+1)^2}{2\alpha}\frac{(1-\delta)}{(1-2\delta)} - 1\right) + 1 = 0. \tag{20}$$

Then, finding the eigenvectors from (19), we write the solution of the system in the form

$$R_k = AR_k^{(1)} + BR_k^{(2)} + CR_k^{(3)} + DR_k^{(4)},$$

where

$$\begin{aligned} R_k^{(1)} &= (1,1); \\ R_k^{(2)} &= \left(k\,(1+\alpha)\frac{2\delta}{1-2\delta} + (1-\alpha),\ k\,(1+\alpha)\frac{2\delta}{1-2\delta} - (1-\alpha)\right); \\ R_k^{(3)} &= \left(-\lambda_3^k\,(\alpha+1)\frac{2\delta}{1-2\delta},\ \lambda_3^{k-1}\left(2\lambda_3^2 - (\alpha+1)\frac{2\,(1-\delta)}{1-2\delta}\,\lambda_3 + 2\alpha\right)\right); \\ R_k^{(4)} &= \left(-\lambda_4^k\,(\alpha+1)\frac{2\delta}{1-2\delta},\ \lambda_4^{k-1}\left(2\lambda_4^2 - (\alpha+1)\frac{2\,(1-\delta)}{1-2\delta}\,\lambda_4 + 2\alpha\right)\right). \end{aligned}$$

Then, we determine the coefficients A, B, C, D from the equations for R_1 and R_{n+1} and write the solutions for $R_k = (r_k^{(1)}, r_k^{(2)})$ $(k = 1, \ldots, n)$:

$$\begin{aligned} r_k^{(1)} &= d\,\frac{x^{2n}(x-\alpha)(1-\alpha x) + x^{2n-k+1}(x-\alpha)(\alpha-1) + x^k(1-\alpha x)(\alpha-1) + (x-\alpha)(1-\alpha x)}{\alpha\,(1-x)\,[x^{2n}(x-\alpha) - (1-\alpha x)]} \\ r_k^{(2)} &= d\,\frac{x^{2n}(x-\alpha)(1-\alpha x) + x^{2n-k+1}(1-\alpha x)(\alpha-1) + x^k(x-\alpha)(\alpha-1) + (x-\alpha)(1-\alpha x)}{\alpha\,(1-x)\,[x^{2n}(x-\alpha) - (1-\alpha x)]}, \end{aligned} \tag{21}$$

where $\underline{x}$ is any root of Eq. (20); $\underline{d}$ is a normalizing multiplier. The solutions $R_{n+k} = (r_{n+k}^{(1)}, r_{n+k}^{(2)})$ $(k = 1, \ldots, n)$ are found from the equations

$$r_{n+k}^{(1)} = r_k^{2}, \quad r_{n+k}^{(2)} = r_k^{(1)}. \tag{22}$$

Making use of expression (15), we arrive at the following for the mathematical expectation of a penalty for the automaton $L_{2n,2}$ with a linear tactic that functions in the medium K:

$$M(L_{2n,2}K) = \frac{1}{2} - \frac{(\alpha-1)^2}{2}\,\frac{\operatorname{ch} ny - 1}{\frac{2n\delta}{1-2\delta}(\alpha+1)^2 \operatorname{ch} ny + (\alpha-1)^2 \operatorname{cth}\frac{y}{2}\operatorname{sh} ny}, \tag{23}$$

where

$$\operatorname{ch} y = \frac{(1+\alpha)^2}{2\alpha}\,\frac{1-\delta}{1-2\delta} - 1, \quad \alpha = \frac{p}{1-p}.$$

Figure 5 shows the curves which characterize the dependence of M on n for various values of δ and p = 0.33.

Figure 6 shows the same graph for a fixed δ = 0.01 and various values of p.

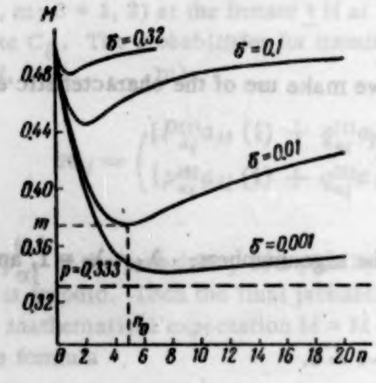


Fig. 5

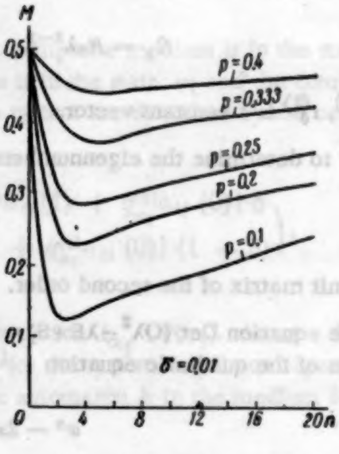


Fig. 6

It is not difficult to prove that $M(L_{2n,2}, K)$ reaches the minimum m for a certain value of the memory capacity = n_0; the cases n = 0 and n = ∞ produce no advantage in comparison to an automaton which does not have expedient behavior.

The presence of this minimum can be explained on the basis of the fact that for a memory capacity which is too small the information on that state of the composite medium in which the automaton functions is not adequately used; for an excessive increase of n there occurs, as it were, an averaging of the statistical properties of the two states of the composite medium. It is natural therefore that as δ decreases we observe an increase in the value of n_0 and the value of m decreases.* For $\delta \to 0$, formula (23) becomes (10) so that $n_0 \to \infty$, $m \to M_{min}$ = min (p, 1−p). As δ increases, the reverse process occurs. Thus, for

$$\frac{1-2\delta}{\delta(1-\delta)} \leqslant \frac{(\alpha+1)^2}{\alpha}$$

the minimal mathematical expectation of the penalty is reached for n = 1.

Formula (21) permits us to construct automata with a linear tactic that have the most expedient behavior in a specified composite medium.

In choosing the memory capacity for such automata it is useful to employ a table which cites the values of n_0 and d = ½ −m for several values of p and δ. In each "cell" of the table we write a pair of numbers: the first number is n_0, and the second number is d.

* Note that based on the meaning of the parameter δ [formula (15)], a decrease in δ is equivalent to increasing the rate at which the automaton operates.

Values of δ

p	0.001	0.010	0.032	0.100	0.320	0.450
0.10	3; 0.396	2; 0.372	2; 0.336	1; 0.256	1; 0.115	1; 0.032
0.20	5; 0.294	3; 0.266	2; 0.223	2; 0.157	1; 0.065	1; 0.018
0.25	6; 0.244	4; 0.212	3; 0.172	2; 0.116	1; 0.045	1; 0.012
0.33	8; 0.158	5; 0.125	3; 0.091	2; 0.055	1; 0.020	1; 0.006
0.40	11; 0.089	6; 0.056	4; 0.037	2; 0.020	1; 0.007	1; 0.002
0.45	16; 0.036	7; 0.017	4; 0.010	2; 0.005	1; 0.002	1; 0.001

Note also that in our case $M_0 = 1/2$, and the difference $d = 1/2 - m$ can serve as a measure for distinguishing between the random media C_1 and C_2 for a finite automaton with a linear tactic.

In conclusion I express my deep appreciation to I. M. Gel'fand, D. S. Lebedev, and O. B. Lupanov for their interest and attention to this paper, and to B. D. Efremov for his great assistance in performing the computations.

LITERATURE CITED

1. M. L. Tsetlin, "Certain problems in the behavior of finite automata," Dokl. AN SSSR, 139, 4 (1961).
2. M.L. Tsetlin, "A matrix method for analyzing and synthesizing electronic-pulse and contact-relay (non-primitive) networks," Dokl. AN SSSR, 117, 6 (1957).
3. M. L. Tsetlin, On Non-Primitive Networks. Symposium "Problemy Kibernetiki" (Problems in Cybernetics) No. 1 [in Russian] (Physics and Mathematics State Press, 1958).
4. N. E. Kobrinskii and B. A. Trakhtenbrot, On Formulating a General Theory of Logic Nets. In the Symposium "Logic Studies" [in Russian] (Academy of Sciences USSR Press, 1959).
5. B. A. Trakhtenbrot, "On operators which are realized in logic nets," Dokl. AN SSSR, 112, 6 (1957).
6. V. I. Shestakov, "An algebraic method for analyzing autonomous two-position relay systems," Avtomatika i Telemekhanika, 15, 2 (1954).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

THE SYNTHESIS OF CONTACT NETWORKS WITH "REAL" CONTACTS*

V. N. Roginskii

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1355-1359, October, 1961

Original article submitted February 24, 1961

The paper presents a method for synthesizing contact networks which do not interrupt operation during transients. A mathematical device based on three-digit logic is cited.

In the process involved in the operation of a multicycle relay network the changes in the states of an individual relay do not occur instantaneously, and in the "transient period" contact circuits may be disrupted due to a non-simultaneous closing and opening of individual contacts of the same relay. An analogous picture holds for the case where several relays change their states during transition between one state of the unit to another (the so-called "competition between relays"). As we know [1-3] the existing contact network algebra, which is based on two-digit Boolean algebra, treats contact networks in static states by eliminating transient periods from the analysis. Recently a number of papers have appeared [4-8] which study the operation of contact networks during transient periods. These papers have chiefly been devoted to analysis. During such periods Gr. Moisil [6] proposed the use of three-digit Lukas'evich logic.

In this paper we demonstrate that this device can be used to synthesize contact networks which do not produce violations during transients.

The static states of the relay, as usual, correspond to 0 and 1; the transient period corresponds to 1/2. The relay contacts at each specified instant can be either open (zero) or closed (1). The states of normally open (a) and normally closed ($\bar{a}$) contacts of the relay A are defined uniquely in static states, but it is impossible to indicate their states during transient periods because the contact may close or open either at the beginning or at the end of the period. Such an indefinite state of the contacts $\underline{a}$ and $\bar{a}$ is denoted by 1/2. One normally open contact plus one

State of relay	0	1/2	1	State of relay	0	1/2	1
a	0	1/2	1	a''	0	1	1
$\bar{a}$	1	1/2	0	$\bar{a}''$	1	1	0
a'	0	0	1	$\hat{a}$	0	1	0
$\bar{a}'$	1	0	0	$\check{a}$	1	0	1

normally closed contact form a switching contact. We shall study two types of switching contacts [9]: the "throw-over" contact (Fig. 1a) for which the normally closed contact first opens when the relay operates, and the normally open contact only closes after this operation has been completed (i.e., there is a time interval during which both contacts are open); and a "transitional" contact (Fig. 1b) for which closure occurs first and is then followed by opening (i.e., there is a period during which both contacts are closed). These contacts are respectively denoted by $a'-\bar{a}'$ and $a''-\bar{a}''$. Moreover, we shall study "temporary-make" (Fig. 1c) and "temporary-break" (Fig. 1d) contacts which close or open the circuit only during transient periods. These contacts are denoted by $\hat{a}$ and $\check{a}$, respectively. The states for all the contacts during different periods of relay operation are characterized by Table 1** (Fig. 2).

* Paper read at the Fourth All-Union Mathematical Conference on July 7, 1961.

** Note that a', $\bar{a}'$, a'' and $\bar{a}''$ correspond to the Lukas'evich modalities [6] νa, ηa, μa, and γa. The symbol $\hat{a}$ for a single relay coincides with the operator "Da" [10], but for multiple-element units they do not correspond to each other.

In order to transform networks with such contacts we shall make use of the three-digit Lukas'evich logic [6] in which the operations of addition (+), multiplication (·), and inversion (N) are defined as follows:

$x+y$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

xy	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

x	0	1/2	1
Nx	1	1/2	0

The letters x, y, and z may correspond to any contact symbols in Table 1 which are treated as variables. Addition and multiplication are interpreted as parallel and series connection of contacts, and inversion is interpreted as finding the circuit with the "reciprocal" structural admittance.

In this algebra, which is a distributive structure, the following relationships are valid:

$$x + x = x, \quad xx = x, \tag{1}$$

$$x + y = y + x, \quad xy = yx, \tag{2}$$

$$x + (y + z) = (x + y) + z, \quad x\,(yz) = (xy)z, \tag{3}$$

$$x\,(y + z) = xy + xz, \quad x + yz = (x + y)\,(x + z), \tag{4}$$

$$x + xy = x, \quad x\,(x + y) = x, \tag{5}$$

$$x + 0 = x, \quad x \cdot 0 = 0, \tag{6}$$

$$x + 1 = 1, \quad x \cdot 1 = x, \tag{7}$$

$$NNx = x, \tag{8}$$

$$N\,(x + y) = Nx \cdot Ny, \quad N\,(x \cdot y) = Nx + Ny. \tag{9}$$

Based on these definitions, we establish the fact that the following relationships hold between functions corresponding to different contacts:

x	a	$\bar{a}$	a'	$\bar{a}'$	a''	$\bar{a}''$	$\hat{a}$	$\check{a}$
Nx	$\bar{a}$	a	$\bar{a}''$	a''	$\bar{a}'$	a'	$\check{a}$	$\hat{a}$

(10)

$$\hat{a} = a'' \cdot \bar{a}'', \quad \check{a} = a' + \bar{a}', \tag{11}$$

$$a + \bar{a} + \hat{a} = 1, \quad a \cdot \bar{a} \cdot \check{a} = 0, \tag{12}$$

$$a'' = a + \hat{a}, \quad \bar{a}'' = \bar{a} + \hat{a}, \tag{13}$$

$$a' = a \cdot \check{a}, \quad \bar{a}' = \bar{a} \cdot \check{a}. \tag{14}$$

Without treating any other of the possible transformations [6] we shall use the algebra cited above to formulate contact networks which include "make" (a), "break" ($\bar{a}$), "throw-over"(a', $\bar{a}'$) and "transitional" (a'', $\bar{a}''$) contacts. If contacts $\underline{a}$ or $\bar{a}$ are present in the structural formula as a result of synthesis, then this means that no requirements are imposed on them with respect to their operation in transient periods.

We shall use the switching table as the basis for synthesis. First we shall study a table which allows transitions only into neighboring states (i.e., in each transition from one cycle to the next a change in the state of just one relay is allowed).

By analogy with the synthesis of networks for static states [1-3] we shall juxtapose the transient period during which the state of a certain relay A changes with a circuit that contains a series temporary-make contact $\hat{a}$ and those contacts of the remaining relays that correspond to their states in the previous and next cycles. Thus, for example, if a network consisting of three relays A, B, and C makes the transition from the state 001 to the state 101, then we shall juxtapose this transition (as well as the reverse transition) with the circuit $\hat{a}\bar{b}c$, which will be closed only for certain transitions.

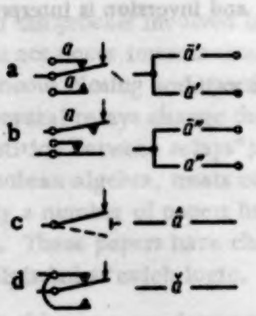


Fig. 1

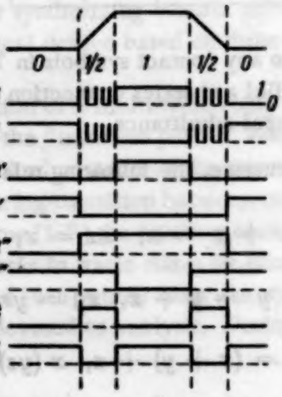


Fig. 2

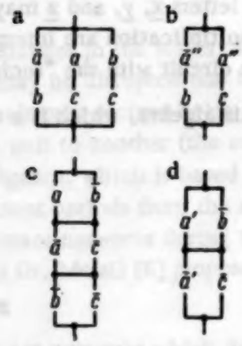


Fig. 3

In order to obtain a circuit which will not be broken during the transient periods it is necessary to sum not only the transfers corresponding to the static network states in which the specified circuit must be closed, but also the expressions corresponding to the transient periods. Under these conditions the general solution [3] can include as conditional components expressions corresponding to all transitions which do not appear during the process of operation of the given network ("unused transitions"). In this case the latter include all double (of the type $\hat{a}\hat{b}c$), triple, etc. transitions. We can assume that all transitions which directly precede or follow the period during which the circuit must be closed are conditional transitions.

As an example we shall study a network with three relays which operate according to the Gray code and in which it is necessary to obtain a circuit that is closed during the cycles 3 to 6. The operation of this network is described by the following switching table:

No. of cycles	0	1	2	3	4	5	6	7	8
A	0	1	1	0	0	1	1	0	0
B	0	0	1	1	1	1	0	0	0
C	0	0	0	0	1	1	1	1	0
f									
No. of states	0	1	3	2	6	7	5	4	0

If we neglect the unused transitions, then for the circuit in $\underline{f}$ the structural admittance will be

$$f = \bar{a}b\bar{c} + \bar{a}b\hat{c} + \bar{a}bc + \hat{a}bc + abc + a\hat{b}c + a\bar{b}c. \tag{15}$$

Performing transformations by factoring and making use of the relationship (12), we obtain

$$f = \bar{a}b(\bar{c} + \hat{c} + c) + \hat{a}bc + ac(b + \hat{b} + \bar{b}) = \bar{a}b + \hat{a}bc + ac. \tag{16}$$

The presence of the term $\hat{a}bc$ indicates that we must create a circuit in the network which closes during the transient period corresponding to operation of the relay A. In order to eliminate $\hat{a}$ we combine the third, fourth, and fifth terms in (15); as a result we obtain

$$f = \bar{a}b + bc + ac. \tag{17}$$

The corresponding network (Fig. 3a) does not produce breaks irrespective of the sequence in which the contacts operate during the transient period.

In this example the unused transitions will be all the double and triple transitions as well as the transitions 000-010, 100-101, 110-111 and 011-001. When we take into account the "extreme" transitions 110-010 and 101-001 the general solution for the circuit f will be

$$\begin{aligned} f = {} & \bar{a}b\bar{c} + \bar{a}b\hat{c} + a\bar{b}c + \hat{a}bc + abc + a\hat{b}c + a\bar{b}c + \\ & + \frac{\bar{a}\hat{b}\bar{c}}{0} + \frac{a\bar{b}\hat{c}}{0} + \frac{ab\hat{c}}{0} + \frac{\bar{a}\hat{b}c}{0} + \frac{\hat{a}b\bar{c}}{0} + \frac{\hat{a}\bar{b}c}{0} + \\ & + \frac{\hat{a}\hat{b}c}{0} + \frac{\hat{a}\hat{b}\bar{c}}{0} + \frac{\hat{a}b\hat{c}}{0} + \frac{\hat{a}\bar{b}\hat{c}}{0} \, \frac{a\hat{b}\hat{c}}{0} + \frac{\bar{a}\hat{b}\hat{c}}{0} + \frac{\hat{a}\cdot\hat{b}\hat{c}}{0}. \end{aligned} \tag{18}$$

Discarding all of the conditional terms which do not result in simplifications we obtain $\hat{a}$:

$$f = a\bar{b} + ac + \hat{a}\frac{b}{c}. \tag{19}$$

Making use of relationship (13), we transform formula (19) so as to eliminate

$$f = \bar{a}b + ac + \hat{a}(b + c) = (\bar{a} + \hat{a})\, b + (a + \hat{a})\, c = \overline{a^*}b + a^*c. \tag{20}$$

This expression corresponds to a network with a transitional contact belonging to the relay A (Fig. 3b).

Inversion in accordance with expressions (8)-(10) leads to networks that have the reciprocal structural admittance not only in their static states but also during their transient periods. For the networks 3a and 3b indicated above we obtain

$$\begin{aligned} N(\bar{a}b + ac + bc) &= (a + \bar{b})(\bar{a} + \bar{c})(\bar{b} + \bar{c}), \\ N(\overline{a^*}b + a^*c) &= (a' + \bar{b})(\bar{a}' + \bar{c}). \end{aligned} \tag{21}$$

Expressions (21) correspond to the networks in Figs. 3c and 3d.

The method cited above can also be used to obtain a circuit that closes only during a certain transition. Thus, for example, if it is required to obtain a circuit g at the instant when the relay A operates for the first time we can write

$$g = \hat{a}b\bar{c} = a^*\overline{a^*}b\bar{c}. \tag{22}$$

Now we shall study the case where the transition is not to a neighboring state (i.e., the case where the successive cycles differ in the states of several relays). Under these conditions we shall assume that the sequence of the lag in the change of state is identical for all the relays. During the transition process it is possible for all combinations of operating and non-operating relay states to appear for relays which change their states.

Thus, for transition of a device with two relays A and B from the state 00 (with the number $i = 0$) to the state 11 ($j = 3$) the transition may take three forms: 00-11 (both relays change their states simultaneously), 00-01-11, and 00-10-11. In order to obtain a circuit (we shall denote it by z_{ij}) which is closed during the entire transition time it is necessary to sum the expressions corresponding to all of the states which may appear; i.e., in this example

$$\begin{aligned} z_{ij} = {} & \hat{a}\hat{b} + \bar{a}\hat{b} + \bar{a}b + \hat{a}b + \hat{a}\bar{b} + a\bar{b} + a\hat{b} = \\ & \hat{a}(\hat{b} + b + \bar{b}) + \hat{b}(\hat{a} + \bar{a} + a) + \bar{a}b + a\bar{b} = \\ & = \hat{a} + \hat{b} + \bar{a}b + \bar{a}b. \end{aligned} \tag{23}$$

By analogous reasoning for the case where s relays $A_1, A_2, \ldots, A_s$ change their states the circuit z_{ij} will have the following form for the transition from the state i (which corresponds to the transfer k_i) to the state j (with a transfer k_j):

$$z_{ij} = \sum_{l=1}^{s} \hat{a}_l + \sum_{l \neq i,j} k_l. \tag{24}$$

The method cited above permits us to take into account not only the static states but also the transient periods during the process involved in synthesizing relay networks; it also makes it possible to determine the necessity of using "throw-over" or "transitional" contacts to assure normal operation of a network during the transient period.

LITERATURE CITED

1. M. A. Gavrilov, The Theory of Contact-Relay Networks [in Russian] (Academy of Sciences USSR Press, 1950).
2. V. N. Roginskii and A. D. Kharkevich, Relay Networks in Telephony [in Russian] (Communications Press, 1955).
3. V. N. Roginskii, Elements of the Structural Synthesis of Relay Control Networks [in Russian] (Academy of Sciences USSR Press, 1959).
4. Gh. Ioanin and Gr. C. Moisil, Asupra functionarii schemelor cu butoni reali. Bul. Stiint. al Acad. R. P. R., cect. st. mat. si fiz., 7, No. 2 (1955).
5. Ya. M. Mekler, "Transient modes in contact-relay networks," Avtomatika i Telemekhanika, 18, 1 (1957).
6. Gr. C. Moisil, "A study of the relay operation of contact-relay networks by means of three-digit logic," Bull. Math. de la Soc. Sci. Math. Phys. de la R. P. R., 1 (49), No. 2 (1957).
7. D. A. Huffman, "The design and use of hazard-free switching networks," J. Assoc. Comput. Machinery, 4, No. 1 (1957).
8. V. N. Roginskii, "The operation of relay networks in transient periods," Avtomatika i Telemekhanika, 20, 10 (1959).
9. Relay Terminology [in Russian] (Academy of Sciences, USSR Press, 1958).
10. A. D. Talentsev, "On the analysis and synthesis of certain electrical networks using special logic operators," Avtomatika i Telemekhanika, 20, 7 (1959).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. ***Some or all of this periodical literature may well be available in English translation.*** **A complete list of the cover-to-cover English translations appears at the back of this issue.**

A LEARNING AUTOMATON OF THE TABULAR TYPE

G. K. Krug and É. K. Letskii

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1360-1366, October, 1961
Original article submitted March 3, 1961

The paper studies the problem of finding the optimal technological mode of operation for complex production processes which do not have a mathematical description. The block diagram is cited for a learning automaton of the tabular type which accumulates and evaluates the information arriving from the object in its natural mode of operation, and then develops recommendations to the operator on the optimum mode for the process. An approximate computation of the learning time is made.

1. Formulating the Control Problem

In the chemical, metallurgical and cement industries there exist very complex processes which can be characterized by the following features: 1) a large number of parameters which affect the course of the process; 2) the presence of uncontrolled external perturbations which affect the process in such a way as to possibly alter the time characteristics of the object; 3) the absence of any information on the form of the function that interrelates the process parameters; in particular, this function may be discontinuous, may have several local extremums, or may have no extremums at all.

We shall introduce the following classification of object parameters (Fig. 1):

1. The set of primary process parameters is denoted by the vector $\mathbf{x}\{x_1, x_2, \ldots, x_m\}$. These are the controlled parameters whose magnitudes depend on external factors and cannot be altered by the operator. The primary parameters include the indices for the input components of the process, certain conditions governing the flow of the process, etc.

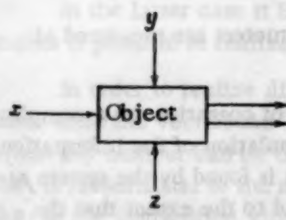


Fig. 1

2. The set of secondary process parameters which characterize the quality of the output product is denoted by the vector $\mathbf{k}\{k_1, k_2, \ldots, k_l\}$.

We shall assume that there is a certain region K^0, where the values of the vector $\mathbf{k}$ satisfy specified requirements governing the quality of the output product.

3. The index of the process effectiveness is denoted by Π. The index can be measured at the output of the object or it can be computed in an arithmetic unit from the known values of the other parameters.

4. The set of controlling parameters which the operator can affect by altering the process in one direction or the other is denoted by the vector $\mathbf{z}\{z_1, z_2, \ldots, z_k\}$.

5. The set of uncontrolled perturbations is denoted by the vector $\mathbf{y}\{y_1, y_2, \ldots, y_n\}$. We call those perturbations uncontrolled for which the magnitudes cannot be measured for some particular reason. In particular, uncontrolled perturbations include variations in the characteristics of the technological equipment due to aging, uncontrolled quality indices of the input components, etc.

The problem of controlling the object reduces to the following: for specified $\mathbf{x}$ and $\mathbf{y}$ it is necessary to determine the value of the vector $\mathbf{z}$ corresponding to the maximum value of the index Π; i.e., it is necessary to find the maximum for the function

$$\Pi = F_{\mathbf{xy}}(\mathbf{z})$$

for the condition $\mathbf{k} \in K^0$ and $\mathbf{z} \in \Gamma_z$, where Γ_z is the region of allowed values of $\mathbf{z}$.

In the case where k is outside the region K^0 it is necessary to determine the value of the vector Δz which returns k to the region K^0.

In view of the absence of complete information on the process and on the equations interrelating the parameters of the object, the operator relies to a considerable extent on his experience and intuition in establishing the technological mode. The effect of subjective factors on the control process often makes the mode of operation far from optimal; this leads to a lowering of the efficiency and the incomplete utilization of the technological equipment, etc.

2. Certain Concepts Concerning the Realization of the Automatic Unit

The problem of designing an automatic unit for achieving an orderly control and finding the optimal technological mode for complex objects is associated with great difficulties. A necessary condition for realizing such a device is the presence of a fleet of primary instruments which provide objective information on the existing factors that determine the technological mode with a sufficient frequency. A preliminary investigation of the process must establish the minimum number of parameters which must be introduced into the control system. For a large number of parameters the introduction of information into the automatic control unit should be realized from a scanning monitor system which scans the process parameters in sequence. For purposes of introducing them into the digital control unit the analog quantities from the outputs of the primary instruments are quantized with respect to level and with respect to time. Time quantization of the input quantities is achieved during the time (t_q) required for a cycle of operation of the scanning monitor system.

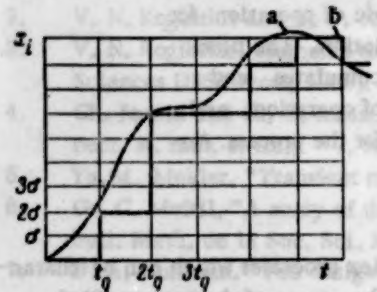


Fig. 2

In Fig. 2 curve a reflects the actual variation of the parameter x_i, and the line b represents the transformed dependence of the parameter x_i on time which was obtained at the output of the scanning monitor system. Here σ is the magnitude of the level quantization, and t_q is the time required for an operating cycle of the system which derives the information.

In studying the problem involving the principle according to which the automatic unit is designed it is necessary to take the following factors into account:

1) the controlled object has a lag t_v determined by the spread between the points at which the primary and secondary information is extracted; here the quantity t_v may reach several tens of minutes;

2) all the primary parameters are measured at the object input; the secondary parameters are measured at the output; the controlling parameters may affect the process at any instant from 0 to t_v.

The device which makes it possible to determine the optimum technological mode of operation for a complex object of the type under study may be a system that performs storage and statistical manipulation of the information arriving from the object when it is in a natural mode of operation. The control algorithm is found by the system as the result of manipulating information from the object. The control algorithm is perfected to the extent that the quantity of manipulated information is increased.

It is evident that at each instant the effectiveness of the control is determined by the degree to which the control algorithm determined by the automaton corresponds to the optimum control method that applies to the state of the object at the instant under study.

When this correspondence is violated it is necessary to adjust the control algorithm; here the algorithm for the actual setting is determined as a result of storing and evaluating information derived from previous settings. A control system of this type can be called a learning system, since the control algorithm which it formulates is perfected with the passage of time to the degree that information arriving from the object is stored and processed.

3. The Block Diagram for a Learning Automaton of the Tabular Type

We shall study the simplest learning system—a learning automaton formulated according to the tabular method [1].

The block diagram for an automaton of the tabular type is shown in Fig. 3. The extraction of information from the object is performed by the scanning monitor system. In extracting the information the system takes into

account the lag and the spread between the points at which the control signals are applied. Information is fed into the automaton at time intervals equal to a cycle of operation t_q for the scanning monitor system. The automaton contains two basic logic blocks and an arithmetic unit AU which is designed for computing the index Π and for calculating the difference $\Delta \mathbf{z} = \mathbf{z}_i - \mathbf{z}_{i-1}$ and the sum $\mathbf{z} + \Delta \mathbf{z}$, where $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$ are the values of the vector $\mathbf{z}$ at the ith and $(i-1)$th instants of information extraction, respectively. The memory of the block 1 is a certain table where it is possible to write up to $\underline{m}$ different vectors $(\mathbf{z}_j, \Pi)$ from the addresses $\mathbf{x}_i$, whereby a new vector is written in the cell with the address $\mathbf{x}_i$ for the condition

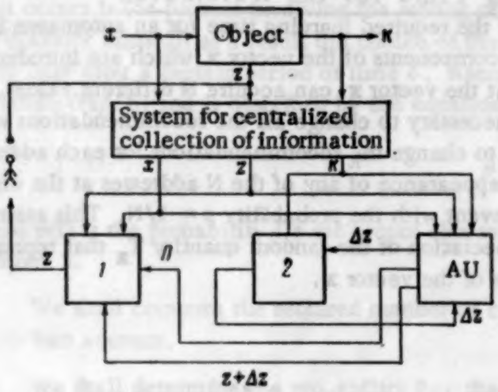


Fig. 3

$$F(\mathbf{z}_j, \mathbf{x}_i, t) = \mathbf{k}_\gamma \in K^0.$$

Determining the mode of operation is achieved cyclically by choosing the vector $\mathbf{z}_j$ corresponding to the maximum value of the index Π from the cell with the address $\mathbf{x}_i$. Thus the mode of operation is found before the information on the secondary parameters is extracted; this is necessary when the values t_γ are appreciable. If the appearance of uncontrolled inputs causes a violation of the process (if in the mode found by the automaton the vector $\mathbf{k}$ is shifted outside the region K^0), then the contents of the memory for block 1 must be changed. This change is performed by means of the block 2, whose memory contains the table in which the vectors $\Delta \mathbf{z}$ are written from the addresses $\mathbf{k}_\gamma$ (the vectors $\Delta \mathbf{z}$ represent quantities by which it is necessary to vary the controlling parameters in order to return the process into the region K^0).

We shall study the problem involving the organization of the memory for block 2.

We shall assume that definite sets of values of the vector $\mathbf{y}$ correspond to a definite probability interrelationship between the magnitudes of the performance deviations and the magnitudes of the variations in the controlling parameters. This interrelationship, for example, may be reflected in the probability sequence governing the use of the different vectors $\Delta \mathbf{z}$ or in the probability frequency governing the use of the vectors $\Delta \mathbf{z}$ for a specified $\Delta \mathbf{k}$.

In the latter case it is possible to obtain a certain distribution of the vectors $\Delta \mathbf{z}$ which for specified conditions makes it possible to realize search in the minimum time.

In order to realize different distributions of a finite number of vectors $\Delta \mathbf{z}$ we can proceed as follows. We shall write the vectors $\Delta \mathbf{z}$ with frequencies proportional to the required probability in the cell with the address $\Delta \mathbf{k}_i$. When any number can be chosen from the cell $\Delta \mathbf{k}_i$ with an equal probability, the probability for choosing each $\Delta \mathbf{z}_j$ is proportional to the number of $\Delta \mathbf{z}_j$ which have been written; therefore it is proportional to the probability of $\Delta \mathbf{z}_j$ in the distribution which is reproduced.

If a variation of the vector $\mathbf{y}$ causes a variation of the frequency with which the specified vector $\Delta \mathbf{z}_j$ is successfully used when a performance deviation $\Delta \mathbf{k}_i$ occurs, then this leads to a variation in the probability with which the vector $\Delta \mathbf{z}_j$ is chosen from the cell $\Delta \mathbf{k}_i$; i.e., it leads to a variation in the distribution of the probabilities with which various $\Delta \mathbf{z}$ are chosen.

A change in the contents of the memory for block 2 changes the algorithm according to which the search for the region K^0 is performed; a change in the contents of the memory for block 1 changes the control algorithm which is formulated for the process. The adaptation of the contents of the memories for both blocks to the state of the object which exists at a specified instant constitutes the learning process in the automaton under study.

The value of the vector representing the controlling parameters $\mathbf{z}_j$ which is found by the automaton is given to the operator in the form of a recommendation with respect to the technological mode of the process. The degree to which the automaton has "learned" can be defined as the ratio between the number of recommendations that have been successfully used by the operator during a specified time interval and the over-all number of recommendations produced.

In principle the automaton under study can also operate without any participation of the operator. For this purpose it is necessary to write $\underline{n}$ vectors Δz of equal probability for each address in the memory of block 2. Since the automaton loop is closed by the object, the automaton sets the memory of block 2 and fills the memory of block 1.

4. The Approximate Estimation of the Learning Time for the Automaton

We shall perform an approximate quantitative estimate of the required learning time for an automaton into which information is introduced at time intervals t_q. Since the components of the vector x which are introduced into the automaton are quantized with respect to level, it follows that the vector x can acquire N different values. Assume that the appearance of the uncontrolled vector y_j has made it necessary to change all the recommendations written in the memory of block 1 of the automaton; here it is necessary to change m_k recommendations for each address x_k. We assume that $k = 1, 2, \ldots, N$. Assume further that the appearance of any of the N addresses at the discrete instants of time $it_q (i = 1, 2, \ldots, S)$ is an independent random event with the probability $p = 1/N$. This assumption is valid for $t_q \gg MT_x$, where MT_x is the mathematical expectation of the random quantity T_x that represents the time interval between the appearance of two different values of the vector x.

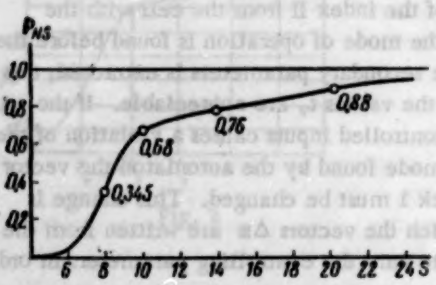


Fig. 4

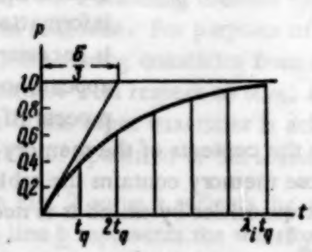


Fig. 5

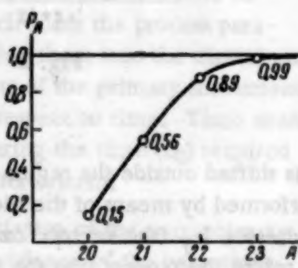


Fig. 6

We shall compute the probability P_{NS} for the appearance of each address x_k no less than m_k times during S cycles:

$$P_{NS} = \frac{\sum_{n_k} C_S^{n_1+m_1} C_{S-n_1-m_1}^{n_2+m_2} \cdots C_{S-n_1-m_1-\ldots-n_{N-1}-m_{N-1}}^{n_N+m_N}}{N^S}. \tag{1}$$

Here N^S is the number of all possible sequences of the values of the vector x at S instants of time it_q.

The numerator of expression (1) is the number of all possible sequences of the values of the vector x in which each value x_k is encountered no less than m_k times; $C_S^{n_1+m_1}$ is the number of combinations of S taken $n_1 + m_1$ at a time; $[n_1, n_2, \ldots, n_k, \ldots, n_N]$ is any combination of the numbers $0, 1, 2, \ldots, \left(S - \sum_{k=1}^{N} m_k\right)$, for which the condition

$$\sum_{k=1}^{N} n_k = S - \sum_{k=1}^{N} m_k$$

is valid.

Simple transformations of the formula (1) yield the following expression for the probability P_{NS}:

$$P_{NS} = \frac{S!}{N^S} \sum_{n_k} \frac{1}{(n_1+m_1)!\,(n_2+m_2)! \ldots (n_k+m_k)! \ldots (n_N+m_N)!}. \tag{2}$$

Figure 4 shows the dependence of P_{NS} on S for $N = 5$ and $m_1 = m_2 = \ldots = m_N = 1$. By specifying the probability P_{NS} it is possible to use formula (2) to determine the number S of cycles t_q required for assuring the appearance of each of the N addresses x_k no less than m_k times with the probability P_{NS}:

$$S = F_1\left(P_{NS},\ N,\ \sum_{k=1}^{N} m_k\right). \tag{3}$$

In using Eq. (3) it is assumed that during the learning process the choice of the correct value of the vector Δz by the operator or by the automaton is achieved with a probability equal to unity at each instant of it_q. In fact, what occurs is the process involving the setting of the memory in block 2 or the process involving the adaptation of the operator himself; as a result the choice of the correct value of the vector Δz occurs with a probability close to unity only after a certain period of time δ. Assume that the process of setting the memory in block 2 is exponential in nature (Fig. 5) and is described by the equation

$$p\_(t) = 1 - e^{-\frac{3t}{\delta}}, \tag{4}$$

where p(t) is the probability for the proper choice of the vector Δz ; δ is the time required for setting the memory in block 2.

We shall compute the required number of cycles t_q for learning when the setting of the memory in block 2 is taken into account.

We shall determine the probability P_{SA} that learning will require A cycles t_q; i.e., we shall determine the probability that for A cycles the vector Δz will be correctly chosen exactly S times. The solution of this type of problem has been treated in [2]. The unknown probability is computed from the formula

$$P_{SA} = \sum_{i=1}^{C_A^S} R_i,$$

where

$$R_i = p(\lambda_1 t_q)\, p(\lambda_2 t_q) \ldots p(\lambda_S t_q)\, [1 - p(\lambda_{S+1} t_q)] \ldots [1 - p(\lambda_A t_q)],$$

C_A^S is the number of combinations in A taken S at a time, $p(\lambda_i t_q)$ is the probability that the correct value of the vector Δz will be chosen at the instant $\lambda_i t_q$; $[\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_A]$ is any transposition of the numbers 1, 2, 3, . . . , A.

The probability P_A that at least A cycles t_q will be required for learning is determined from the formula

$$P_A = \sum_{j=0}^{A-S} \sum_{i=1}^{C_A^{S+j}} R_i. \tag{5}$$

Figure 6 shows the dependence of P_A on A for S = 20 and $\delta = 0.3\ St_q$.

Specifying the probabilities P_A, it is possible to determine the required number A of learning cycles using formula (5):

$$A = F_2\,[P_A,\ S,\ p(t)]. \tag{6}$$

A simultaneous study of expressions (3) and (6) permits us to compute the over-all learning time with the probability $P = P_{NS}P_A$ when the time taken to set the memory of block 2 is taken into account:

$$T_L = F_2\left[P_A,\ F_1\left(P_{NS},\ N,\ \sum_{k=1}^{N} m_k\right),\ p(t)\right] t_q. \tag{7}$$

In the example treated above, when N = 5 and $m_1 = m_2 = \ldots = m_N = 1$, and the probabilities $P_{NS} = 0.88$ and $P_A = 0.997$ are specified, it is possible to state that $T_L \le 23\ t_q$ with the probability P = 0.87.

For successful operation of the automaton it is necessary to satisfy the inequality

$$T_L \leqslant \frac{1}{\beta} T_{y\min} \quad \text{for } \beta > 1, \tag{8}$$

where $T_{y\min}$ is the minimum time between substantial variations of the vector representing the uncontrolled inputs that is manifested with a sufficient degree of reliability.

A learning automaton of the tabular type which is designed according to the scheme described above can be used both for ordering the process involved in the control of operating objects and for formulating technological operating-mode charts for objects that are being made operational.

LITERATURE CITED

1. A. V. Netushil, G. K. Krug, and É. K. Letskii, The Use of Learning Automatons in the Automatization of Complex Production Processes [in Russian] (Report on NIR, Moscow Power Institute, No. 20860, 1960).
2. E. S. Venttsel', Probability Theory [in Russian] (Physics and Mathematics State Press, 1958).

A FLUCTUATING-NOISE GENERATOR FOR INVESTIGATING INFRALOW-FREQUENCY CONTROL OBJECTS

Yu. M. Bykov

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1367-1372, October, 1961
Original article submitted May 30, 1961

The paper describes a white-noise generator which is designed for use in statistical investigations of infralow-frequency objects.

The principle governing the operation of the generator is based on using fluctuations in a stream of particles from an ultralow-power source of radioactive radiation.

The use of statistical methods to investigate automatic control systems is in a number of cases associated with the use of noise-signal generators.

The usual requirement applied to noise generators is uniformity of the spectral noise power in a specified frequency band and a definite law governing the distribution of the signal amplitudes. An important requirement is a stationary noise signal and the absence of any parasitic regular components.

We are thoroughly acquainted with noise generators that are used in the audiofrequency and radiofrequency ranges and use a gas-discharge device (for example, a thyratron) or a vacuum device (for example, a noise diode of the type "DSh-2," etc.) as the primary source of the noise emf. The spectrum for the noise emf generated by such sources is usually sufficiently uniform over a wide frequency band. Such sources are low-power devices. In a number of cases it is necessary to use noise-signal generators operating in the ultralow-frequency range when statistical investigations of automatic control systems are made. The generation of noise emfs in this range by means of the primary sources indicated above is an extremely complex engineering problem, since a portion of the energy from the noise-signal source that is confined to the required low-frequency band represents an infinitesimal part of the overall energy in the noise signal produced by the primary source. In this case the use of amplification of the order of 10^4-10^6 produces distortion of the energy distribution of the original signal over the spectrum; it also introduces an additional nonstationary factor and parasitic regular components.

It is especially difficult to obtain a stationary noise signal at a sufficiently high level with the required statistical characteristics in the infralow-frequency range $1 \cdot 10^{-3}$ to 1 cps. It should be noted that this range is characteristic for a large number of automatic control objects, in particular for the majority of industrial objects.

The literature describes several generator networks which produce low-frequency noise signals [1-3]. The essence of the operation of these generators resides in using a band-pass filter to isolate a sufficiently uniform sector of the noise generator spectrum (for example, the spectrum produced by a gas-discharge device) at a frequency of the order of 1 to 10 kc and then converting this isolated noise into low-frequency noise by heterodyning. The structure of a generator based on such a method is shown in the block diagram of Fig. 1, which is described in [3]. Such methods do not eliminate the necessity of employing large gains; this, as we have pointed out above, is undesirable. In [2], for example, the degree to which the noise signal is stationary is improved by making use of a so-called "noise AGC" circuit which is sometimes used in radio receivers. The use of AGC greatly complicates the noise generator circuit.

The generators described in [1-3] produce noise with a uniform distribution over the spectrum in a band of the order of 0 to 30 cps. In order to obtain noise in a band covering lower frequencies (in a band ranging from 0 to 3 cps) the method [4] based on using fluctuations of an electrical resistance formed by overflowing metal spheres between the conducting axis of a rotating drum and its conducting cylindrical surface has been proposed. The inconvenience of using such a generator resides in the presence of mechanical and electromechanical elements, as well

as in the dependence of the statistical characteristics of the generated noise on the structural features of the device or their variations.

The method described below for obtaining a noise signal in the infralow-frequency range is based on the use of a radioactive-particle counter and permits us to use a simple circuit to obtain a noise signal with an adequate level (dispersion) and the uniform spectral density in the required frequency band. The law governing the amplitude distribution is practically the same as the normal law.

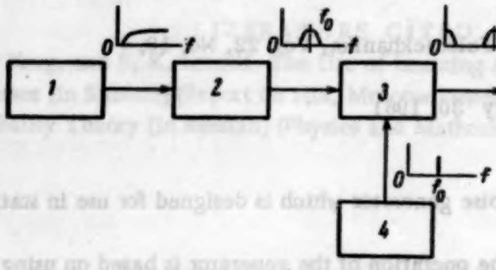


Fig. 1. Block diagram for obtaining low-frequency noise by the heterodyning method: 1) Audiofrequency noise generator; 2) band-pass filter; 3) mixer; 4) heterodyne; 5) low-pass filter; 6) output.

In this case we use the fluctuations in the radioactive decay process that occurs in an ultralow-power source as the primary noise signal.*

As we know, such processes are characterized by the Poisson distribution law

$$P(k) = \frac{\lambda t}{k!} e^{-\lambda t}. \tag{1}$$

The quantity λt which appears in expression (1) is the average number of particles produced during the time interval t.

The pulses obtained at the output of the gas-discharge counter can be converted by means of a multivibrator with two stable states into a so-called random telegraph signal; this signal is a voltage that acquires one of two possible values (u or $-u$) in random fashion (cf. Fig. 2).

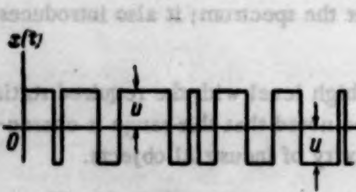


Fig. 2. Random telegraph signal.

From expression (1) it is not difficult to obtain the expression for the spectral density $G(\omega)$ of a random telegraph signal in terms of the correlation function for the process (cf., for example, [5, 6]):

$$G(\omega) = \frac{4u^2}{\pi\lambda} \frac{1}{1+\left(\frac{\omega}{\lambda}\right)^2}. \tag{2}$$

The graph for the spectral density $G(\omega)$ is shown in Fig. 3a. From expression (2) and the graph in Fig. 3a it is evident that in the low-frequency range it is possible to obtain a sufficiently uniform energy distribution. The uniformity of the spectral density $G(\omega)$ can easily be seen to depend on the relationship between λ and ω. By varying λ it is possible to obtain the required uniformity of the curve $G(\omega)$ in a pass-band with the specified upper frequency limit ω_{hi}. In the majority of practical cases it is possible to use the simplified expression for the spectral density in the low-frequency range:

$$G(\omega) \approx \frac{4u^2}{\pi\lambda}. \tag{3}$$

* For example, such a source may consist of a granule of glowing material which is used to coat the numbers on the scales of instruments, clocks, etc.

A precise estimate of the required value of λ in the frequency range with the specified upper frequency limit ω_{hi} and a definite degree of nonuniformity $\underline{n}$ can be obtained from the relationship

$$\lambda = \omega_{hi} \sqrt{\frac{1}{n} - 1} \approx \omega_{hi} \frac{1}{\sqrt{n}}, \tag{4}$$

where ω_{hi} is the upper frequency limit for the required noise band, $\underline{n}$ is the degree of nonuniformity of the spectral density:

$$n = \frac{G(0) - G(\omega_{hi})}{G(0)}. \tag{5}$$

Expression (4) makes it possible to estimate the value of the upper frequency limit ω_{hi} for a specified degree of nonuniformity $\underline{n}$ and an experimentally determined value of λ. The method for determining λ is indicated below.

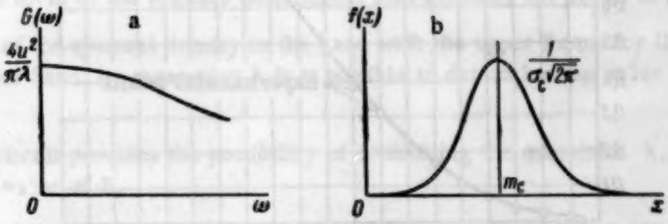


Fig. 3. Graph for the spectral density $G(\omega)$ and the probability density $f(x)$ of the generated noise.

The block diagram of the generator is shown in Fig. 4. As the counter 1 we can use one of the types of standard halogen radioactive-particle counters (for example, the "STS-5" model). The multivibrator 2 is used to convert the chaotic pulse train arriving from the output of the counter 1 into a random telegraph signal (Fig. 2). The time constant for the low-pass filter 3 is chosen from the condition

$$T_f \gg \frac{1}{\lambda}, \tag{6}$$

where λ is the mean pulse frequency required to assure the specified $\underline{n}$ in a pass-band with the upper frequency limit ω_{hi}. The buffer stage 4 is designed to eliminate the effect of the load on the mode of operation of the generator.

The measurement of λ is performed by switching the multivibrator input (and the input of the integrating circuit 6 with the dc meter 7) to a calibrated pulse generator 5. The value of the measured λ corresponds to the reading on the generator scale (in cps) for which the dc component of the voltage at the integrator output (measured by the meter 7) is equal to the dc component obtained when the integrator is connected to the output of the particle counter 1. A variation of λ is easily achieved by controlling the position of the β-particle source relative to the counter frame.

The generator circuit contains two dual triodes (of the bantam or ultraminiature type). One of these forms the multivibrator with two stable states that is triggered by the pulses from the gas-discharge counter. One-half of the other dual triode forms the buffer stage (a cathode follower). The second half of this tube is used as a charging diode in the integrating circuit 6. When it is necessary to control or subtract the dc component we use a divider which is fed from a common plate voltage source.

The supply source for the generator consists of the simplest rectifier circuit, which is used for supplying the plate circuits, a step-down winding for supplying the filament circuits, and a voltage doubler rectifier circuit for supplying the counter. The generator operates from a 220/127 v, 50 cps line.

The level of the noise signal (mean-square deviation) can be controlled over the range 0 to 6 v by varying the amplitude of the pulse voltage taken from the output of the multivibrator tube.

In performing an experimental investigation of the generator operation we shall devote our major attention to clarifying the law governing the noise amplitude distribution. Based on an analysis of the principle governing the

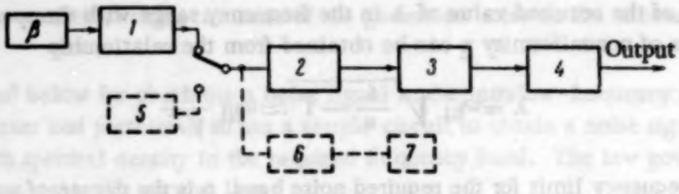


Fig. 4

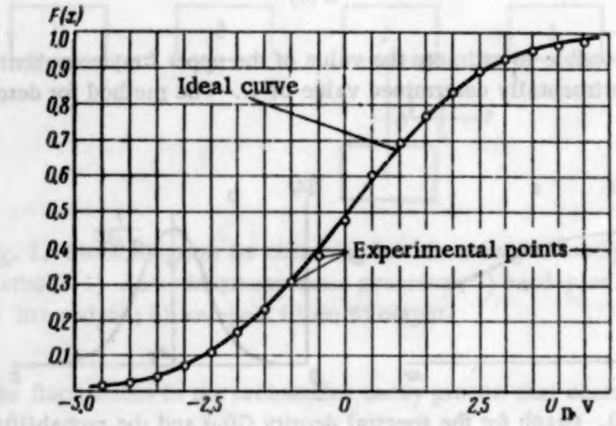


Fig. 5

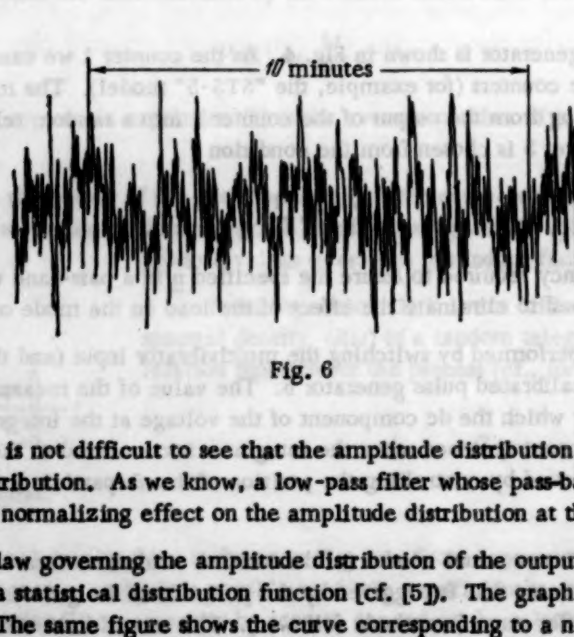


Fig. 6

operation of the generator, it is not difficult to see that the amplitude distribution of the signal before the low-pass filter differs from normal distribution. As we know, a low-pass filter whose pass-band is appreciably less than the input noise band has a strong normalizing effect on the amplitude distribution at the output (cf., for example, [7]).

In order to clarify the law governing the amplitude distribution of the output signal we investigated experimental data and formulated a statistical distribution function [cf. [5]). The graph for the statistical distribution function is shown in Fig. 5. The same figure shows the curve corresponding to a normal distribution law with a dispersion equal to the statistical dispersion of the investigated noise. The closeness of the curves proves that the normalizing effect of the low-pass filter is adequate for practical purposes. When necessary, inequality (8) can be strengthened. Thus, the probability density of the generated noise can be approximated with a sufficient accuracy by the expression for the normal (Gaussian) distribution law

$$f(x) = \frac{1}{\sigma_c \sqrt{2\pi}} e^{\frac{-(x-m_c)^2}{2\sigma_c^2}}, \qquad (7)$$

where m_c is the mathematical expectation; σ_c is the mean-square deviation.

The graph of the function f(x) is shown in Fig. 3b. As an example of the resulting signal Fig. 6 shows one of its realizations.

The resulting data makes it possible to draw the following conclusions concerning the proposed method of obtaining infralow-frequency noise.

1. The generated noise signal has a uniform spectral density $G(\omega)$ which is determined by the expression (3) in the infralow-frequency range beginning at zero frequency.

2. The probability distribution for the noise amplitude can be approximated by the normal distribution law with a sufficient accuracy for practical application.

3. It follows that the noise signal is stationary from the properties of radioactive decay. As we know, any physical (or chemical) conditions cannot affect the decay rate. The stationary nature of the primary noise process is not violated in any other elements of the generator circuit in view of the appreciable level of the primary noise signal and the absence of amplifier elements in the circuit.

4. The appreciable level of the primary noise signal also excludes the effect of parasitic regular components.

5. The uniformity of the spectral density in the band with the upper frequency limit ω_{hi} is easily controlled by varying λ. On the other hand, by measuring λ it is possible to determine the value of ω_{hi} for a specified value of $\underline{n}$.

6. The generator circuit provides the possibility of controlling the quantities λ, m_c, σ_c, and also provides a circuit for measuring the value of λ.

7. The simplicity of the generator circuit assures reliable operation without any need for setting or regulating during the operating process.

LITERATURE CITED

1. R. R. Bennett and A. S. Fulton, The Generation and Measurement of Low Frequency Random Noise. Journ. Appl. Phys. Vol. 22, No. 9, September (1951).
2. H. Becher, R. R. Bennett, and H. Low, Stabilized Noise Source for Air-Weapons Design. Electronics, 27, No. 7 (1954).
3. Yu. M. Romanovskii, "A low-frequency noise generator," Pribory i Tekhnika Éksperimenta, No. 4 (1958).
4. A. M. Petrovskii, "A fluctuation-noise generator for infrasonic frequencies," Avtomatika i Telemekhanika, 14, No. 4 (1953).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

PROPORTIONAL-INTEGRATION AND PROPORTIONAL-DIFFERENTIATION ELEMENTS BASED ON RC CIRCUITS WITH MAGNETIC AMPLIFIERS

O. M. Minina, V. S. Serzhers, and G. V. Subbotina

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1373-1385, October, 1961
Original article submitted February 14, 1961

The paper studies proportional-integration and proportional-differentiation elements based on RC circuits with magnetic amplifiers; these elements are used in controllers. A method is given for choosing the parameters of the elements. Data based on an experimental investigation of highly stable proportional-integration and proportional-differentiation elements is cited.

In order to design proportional-integration (PI) and proportional-differentiation (PD) controller elements it is customary to make use of RC circuits with electronic amplifiers. The reliability of PI and PD elements of controller units can be appreciably increased by using magnetic amplifiers.

PI and PD elements are defined as elements with the transfer functions

$$W_{\mathrm{PI}}(p) = k + \frac{1}{T_i p}, \tag{1a}$$

and

$$W_{\mathrm{PD}}(p) = k(1 + T_d p), \tag{1b}$$

where k is a proportionality coefficient, T_i is the integration time constant, T_d is the differentiation time constant.

Proportional-Integration Elements Based on RC Circuits with Magnetic Amplifiers

Figure 1a shows the block diagram of a magnetic amplifier with an RC circuit. The control winding W_c is supplied with dc current i_c. A load resistance R_L is connected in series with the operating windings W_L (these are not shown in Fig. 1a) that are supplied with alternating current. In order to obtain integrating elements based on magnetic amplifiers a capacitance C (R_{lk} is the leakage resistance of the capacitor) and a resistor R_{nfb} are connected in the circuit of the negative feedback winding W_{nfb}. An increase in the amplifier gain is achieved by introducing positive feedback through the appropriate winding W_{pfb}.

From an analysis of the block diagram shown in Fig. 1 we obtain an expression in operator form which relates the input voltage to the output voltage:

$$\bar{u}_{\mathrm{out}} = \bar{u}_{\mathrm{in}} k_1 \frac{R_L + R_3}{R_c + R_1} \frac{b_0 p^2 + b_1 p + b_2}{a_0 p^2 + a_1 p + a_2}. \tag{2}$$

Here

$$b_0 = T_{fb}\left(\frac{T_3}{k_3} - \frac{T_2}{k_2}\right), \quad b_1 = T_{fb} + \frac{T_3}{k_3} + \frac{T_{fb}}{T_{lk}}\left(\frac{T_3}{k_3} - \frac{T_2}{k_2}\right), \quad b_2 = \frac{T_{fb}}{T_{lk}} + 1,$$

$$a_0 = T_1(T_{fb} + T_L) + T_3 T_L\left(1 + \frac{k_2}{k_3}\right) + T_2 T_{fb}\left(1 + \frac{k_3}{k_2}\right),$$

$$a_1 = T_{fb} + T + T_1 + \frac{T_L}{T_{lk}}\left[T_1 + T_3\left(1 + \frac{k_2}{k_3}\right)\right] + \frac{T_{fb}}{T_{lk}} T_2\left(1 + \frac{k_3}{k_3}\right),$$

$$a_2 = \frac{T}{T_{lk}} + 1,$$

where

$$k_1 = \frac{W_{\mathrm{c}} k_I}{W_{\mathrm{L}}\left(1 - \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the current gain referred to the control winding of the magnetic amplifier MA;

$$k_2 = \frac{W_{\mathrm{nfb}}\ k_I}{W_{\mathrm{L}}\left(1 - \frac{W_{\mathrm{nfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the current gain referred to the negative feedback winding;

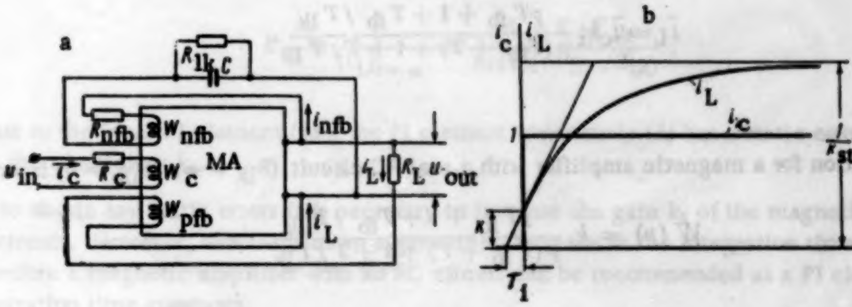


Fig. 1

$$k_3 = \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}\left(1 - \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the current gain referred to the positive feedback winding;

$$T_1 = \frac{L_1}{(R_{\mathrm{c}} + R_1)\left(1 - \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the time constant for the control winding circuit (L_1 and R_1 are the inductance and resistance of the control winding);

$$T_2 = \frac{L_2}{(R_{\mathrm{nfb}} + R_2)\left(1 - \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the time constant of the negative feedback winding circuit (L_2 and R_2 are the inductance and resistance of the negative feedback winding);

$$T_3 = \frac{L_3}{(R_{\mathrm{L}} + R_3)\left(1 - \frac{W_{\mathrm{pfb}}\ k_I}{W_{\mathrm{L}}}\right)}$$

is the time constant of the positive feedback winding circuit (L_3 and R_3 are the inductance and resistance of the positive feedback winding);

$$T_L = R_L C,\ T_{fb} = (R_{nfb} + R_2)C,\ T = \frac{R_L C\left(1 + \frac{W_{nfb}}{W_L}k_f\right)}{1 - \frac{W_{pfb}}{W_L}k_f}$$

are the time constants of the load circuit; $T_{lk}=R_{lk}C$ is the time constant of the capacitor leakage circuit.

Usually the time constants of PI and PD controller elements in units designed for the automatization of technological processes are equal to several seconds; therefore $T_1, T_2, T_3 \ll T$ and T_{fb} ($T_{lk} \gg T$ and T_{fb}). In accordance with this formula (2) can be simplified considerably:

$$\bar{u}_{out} = \bar{u}_{in} k_1 \frac{R_L + R_3}{R_c + R_1} \frac{pT_{fb} + 1 + T_{fb}/T_{lk}}{p(T_{fb} + T) + 1 + T/T_{lk}}. \tag{2a}$$

The dependence of the load current on the current in the control winding becomes

$$\bar{i}_L = \bar{i}_c k_1 \frac{pT_{fb} + 1 + T_{fb}/T_{lk}}{p(T_{fb} + T) + 1 + T/T_{lk}} \tag{3}$$

under these conditions.

The transfer function for a magnetic amplifier with a real RC circuit ($R_{lk} \neq \infty$, $R_{nfb} \neq 0$) is written as

$$W(p) = k_1 \frac{pT_{fb} + 1 + T_{fb}/T_{lk}}{p(T_{fb} + T) + 1 + T/T_{lk}}. \tag{4}$$

In order to lower the effect of capacitor leakage on the operation of units with RC circuits we attempt to choose T_{fb} and $T \ll T_{lk}$.

It should be noted that the magnitude of leakage resistance R_{lk} depends on the temperature of the ambient medium; therefore as the temperature increases T_{lk} decreases. This can lead to instability in the operation of units with RC circuits.

When the resistance of the negative feedback winding is appreciable or when there is a series resistance in the feedback circuit the magnetic amplifier with an RC circuit can be treated as a proportional-integration element (a PI element) since $T_{fb} < T + T_{fb}$.

We shall determine the parameters of this element on the basis of analyzing the transient responses for a unit step variation of the control current (Fig. 1b for $t > 0$, $i_c = 1$). Here the load current is determined from the expression

$$i_L = k_1 \frac{T_{fb}}{T_{fb} + T}\left[1 + \frac{T_{fb}/T_{lk} + T/T_{fb}}{1 + T/T_{lk}}\left(1 - \exp\left(-\frac{1 + T/T_{lk}}{T_{fb} + T}\right)\right)\right] \tag{5}$$

We shall define the proportionality coefficient $\underline{k}$ of the element as the ratio between the current in the load of the element and the control current at the instant when a step variation of the control occurs (for $t = 0$):

$$k = \frac{[i_L]_{t=0}}{[i_c]_{t=0}} = k_1 \frac{T_{fb}}{T_{fb} + T}. \tag{6}$$

The integration time constant T_i can be characterized by a quantity which is the reciprocal of the rate of rise of the load current at the instant $t = 0$ when the current in the control winding varies stepwise:

$$T_i = \frac{1}{\left[\frac{di_L}{dt}\right]_{t=0}} = \frac{(T_{fb} + T)^2}{k_1\left(\frac{T_{fb}^2}{T_{lk}} + T\right)}. \tag{7}$$

TABLE 1

Type of PI element	Parameter of PI elements		
	k	T_i	k_{st}
With an ideal RC circuit	k_3	$T\frac{1}{k_1k_4}$	$\frac{1}{k_3+k_1k_4}$
With an RC circuit when capacitor leakage is taken into account	k_3	$T\frac{1}{k_1k_4}$	$\frac{1+T/T_{1k}}{k_3(1+T/T_{1k})+k_1k_4}$

The statistical error coefficient k_{st} shall be defined as the quantity which is the reciprocal of the value of load current in a steady-state mode for stepwise variation of the control current:

$$k_{st}=\frac{1}{[i_L]_{t\to\infty}}=\frac{1+T/T_{1k}}{k_1(1+T_{fb}/T_{1k})}. \tag{8}$$

In contrast to the ideal PI element (1a), the PI element under study (4) has a static error that is inversely proportional to the coefficient k_1.

In order to obtain low static errors it is necessary to increase the gain k_1 of the magnetic amplifier with respect to the control circuit. However, this leads to an appreciable decrease in the integration time constant of the PI element. Therefore a magnetic amplifier with an RC circuit can be recommended as a PI element only for systems with low integration time constants.

When there is no series resistance in the feedback circuit ($T_{fb} \ll T$) and $R_{1k}=\infty$, the transfer function for the magnetic amplifier is written as

$$W(p)=k_1\frac{1}{Tp+1}.$$

A magnetic amplifier with an ideal RC circuit connected as shown in Fig. 1a is an aperiodic section; here $k = k_1$ is the transfer coefficient for the section and T is the time constant for the section.

A PI element can be designed using two magnetic amplifiers (Fig. 2). The control signal in this circuit is applied simultaneously to the inputs of the magnetic amplifier with the RC circuit (MA_1) and a summing magnetic amplifier (MA_2). The output of the magnetic amplifier with the RC circuit is connected into the second control circuit of the summing magnetic amplifier.

The transfer function for this element will be given by

$$W(p)=k_3+\frac{k_1k_4}{Tp+1}=\frac{k_3Tp+k_3+k_1k_4}{Tp+1}. \tag{9}$$

If we take into account the leakages in the RC circuit of the amplifier MA_1, then the transfer function for the PI element will be

$$W(p)=\frac{\frac{T}{1+T/T_{1k}}k_3p+k_3+\frac{k_1}{1+T/T_{1k}}k_4}{\frac{T}{1+T/T_{1k}}p+1}, \tag{10}$$

where k_1 and T represent the gain and time constant of the magnetic amplifier with the RC circuit, k_3 and k_4 represent the gains of the summing magnetic amplifier.

It is obvious that

$$\frac{k_3T}{k_3+k_1k_4}<T \text{ and } \frac{k_3T}{k_3(1+T/T_{1k})+k_1k_4}<T,$$

and therefore the element under study (Fig. 2) can be used as a PI element in controllers.

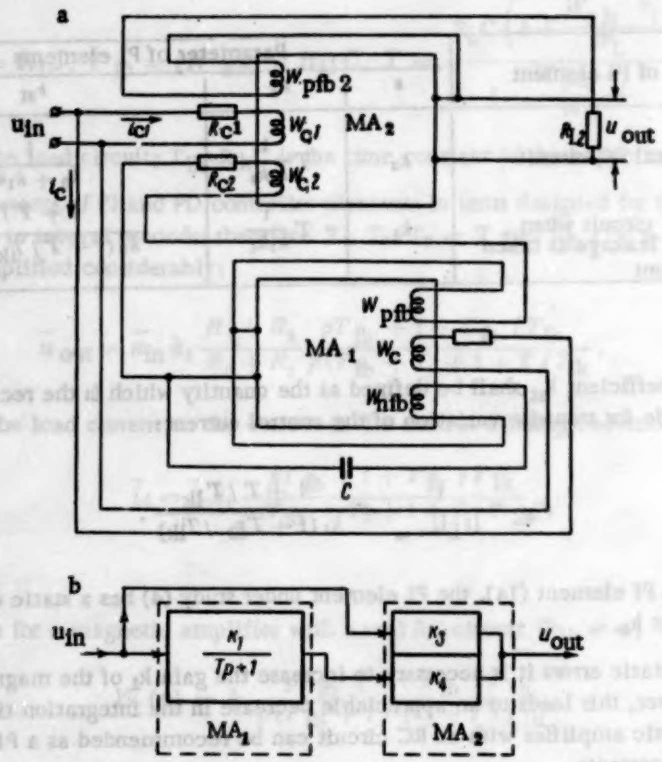


Fig. 2. a) Circuit diagram of PI element in magnetic amplifiers, b) block diagram of PI element.

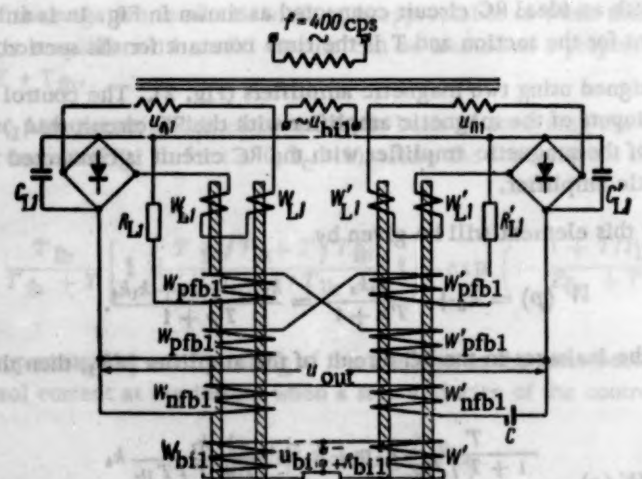


Fig. 3. Circuit for the magnetic amplifier with an RC circuit.

The parameters of PI elements connected as per the circuit in Fig. 2 are shown in Table 1.

From Table 1 it is evident that the leakages of the RC circuit in the amplifier MA_1 of the PI element do not affect the parameters k and T_1. In order to reduce the static error in a specified PI element it is necessary to increase the gains (k_1, k_3, k_4) of the magnetic amplifiers with respect to their control circuits. This can be achieved without reducing the integration time constant if the coefficient k_3 is increased. The integration time constant can be increased in comparison to the time constant of an aperiodic section (for $k_1k_4 < 1$). In the magnetic amplifier MA_1 it is also expedient to choose $T \ll T_{1k}$.

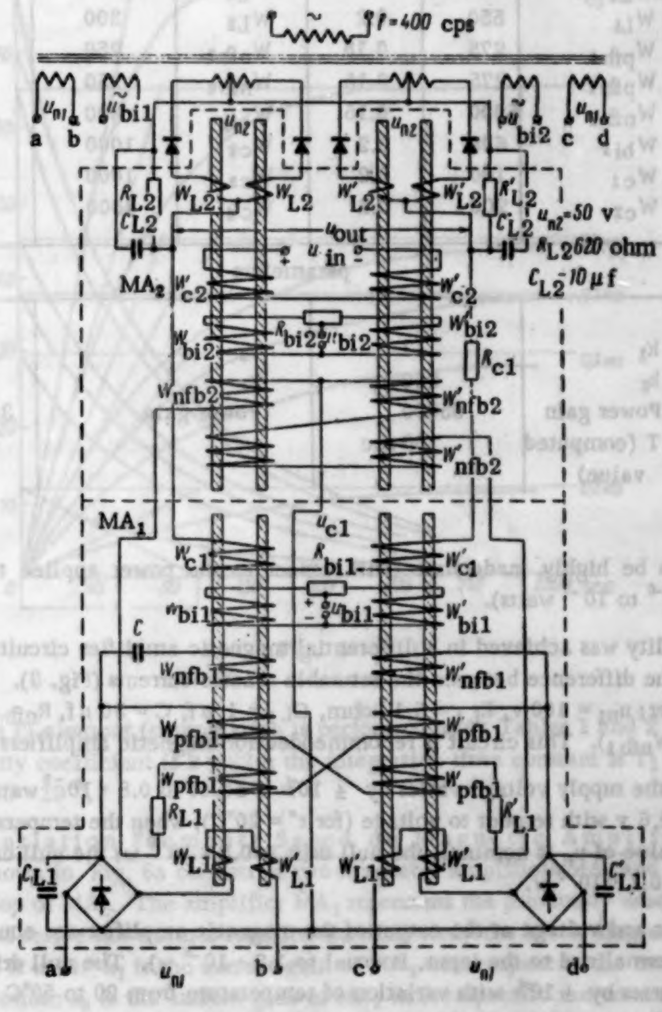


Fig. 4. Circuit diagram of a PD element.

Thus the latter circuit can be recommended as a PI element for systems with an appreciable integration time constant.

The basic requirement which is imposed on magnetic amplifiers with RC circuits is the necessity of achieving an appreciable time constant T for a high null stability.

The time constant depends on a load resistance R_L, the capacitance of the capacitor C, and the winding ratios W_{nfb}/W_L and W_{pfb}/W_L. The assurance of high null stability for the amplifier when the positive feedback coefficient is large presents well-known difficulties.

We performed experimental investigations of the stability of differential magnetic amplifier circuits with simultaneous use of internal and external positive current feedback based on the current in each saturable reactor, and amplifiers with simultaneous utilization of internal positive current feedback with respect to the current in each saturable reactor and external positive voltage feedback. When the supply voltage varies by $\pm 10\%$ the null stability

TABLE 2

Magnetic amplifier with RC circuit			Summing magnetic amplifier		
		winding data			
	turns	diameter, mm		turns	diameter, mm
W_{L1}	550	0.2	W_{L2}	300	0.35
W_{L1}	550	0.2	W_{L2}	300	0.35
W_{pfb1}	275	0.15	W_{nfb2}	250	0.2
W_{pfb1}	275	0.15	W_{nfb2}	250	0.2
W_{nfb1}	5000	0.15	W_{bi2}	500	0.2
W_{bi1}	600	0.2	W_{c2}	1000	0.2
W_{c1}	100	0.2	W_{c3}	1000	0.2
W_{c2}	200	0.2	W_{c4}	1000	0.2
		parameters			
k_1	13		k_3		220
k_2	650		k_4		56
Power gain	95000		Power gain		380000
T (computed value)	100 sec				

of such circuits proved to be highly inadequate (with respect to the power applied to the magnetic amplifier input the null drift was 10^{-6} to 10^{-7} watts).

The maximum stability was achieved in a differential magnetic amplifier circuit which used external positive feedback with respect to the difference between the saturable reactor currents (Fig. 3). We shall indicate certain parameters of this amplifier: u_{n1} = 100 v, R_{L1} = 5.1 kohm, C_{L1} = 1 μf, C = 30 μf, R_{nfb} = 350 ohm (the internal resistance of the winding W_{nfb1}). This circuit is recommended for magnetic amplifiers with RC circuits.

The null drift when the supply voltage varies by ± 10% is equal to $0.8 \cdot 10^{-9}$ watts with respect to the power applied to the input; it is 0.6 v with respect to voltage (for t° = 20°C); when the temperature of the medium varies from 20 to 50°C and the value of u_n is nominal, the null drift is $0.5 \cdot 10^{-9}$ w; the null drift for continuous operation over a period of 8 hours is $0.1 \cdot 10^{-9}$ w.

The maximum power and voltage at the output of the magnetic amplifier are equal to 2.1 w and 100 v (the maximum output power, normalized to the input, is equal to $1.2 \cdot 10^{-5}$ w). The null drift of the MA is insignificant when the supply voltage varies by ± 10% with variation of temperature from 20 to 50°C or for continuous operation of the unit over a period of 8 hours.

In a summing magnetic amplifier (MA_2 in Fig. 4) we use a differential circuit with internal positive feedback with respect to the current through each saturable reactor. The null drift, expressed in terms of the power normalized to the input, is $0.5 \cdot 10^{-8}$ w when the supply voltage varies by ± 10%; it is 1 v at the output with respect to voltage (when the temperature of the ambient medium is 20°C). When the temperature of the ambient medium varies from 20 to 50°C the null drift is $0.15 \cdot 10^{-8}$ w (for a nominal supply voltage). The null stability for continuous operation over a period of 8 hours is 10^{-9} w. The maximum load power is 0.66 w for a voltage of 20 v.

The magnetic amplifier with an RC circuit and the summing amplifier have wound toroidal cores made of the alloy "79NMA" (ribbon thickness 0.1 mm; core dimensions 50 x 32 x 10). We used wire of the type "PÉLShO" for the windings. Rectifying silicon diodes of the type "D204" were used in the circuits.

The data for the amplifier windings is given in Table 2 in accordance with the notation used in the figure.

Figure 5 shows the transient responses for the magnetic amplifier with the RC circuit shown in Fig. 3. The descending curves correspond to a stepwise variation of i_{c_1} from the corresponding value to zero. From Fig. 5 it follows that for the indicated circuit parameters a time constant of $T \approx 100$ sec is obtained (in the linear range of operation). The difference between the experimental and computed values of the time constant T is explained by deviations of the actual values of R_L and C from the nominal value.

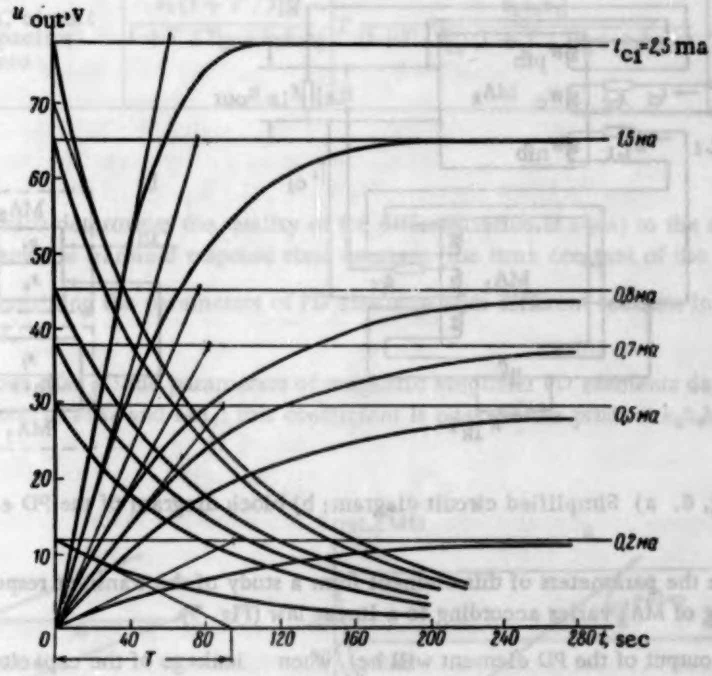


Fig. 5

The parameters of the PI element (computation is performed using Tables 1 and 2 for $T_{1k} \gg T$) are the following: the proportionality coefficient is k = 220; the integration time constant is $T_1 = 30$ sec; * and the static error coefficient is $k_{st} = 1.05 \cdot 10^{-3}$.

Proportional-Differentiation Elements Based on Magnetic Amplifiers

The block diagram shown in Fig. 6a consists of two magnetic amplifiers MA_1 and MA_2; here MA_1 is connected into the feedback loop of MA_2. The amplifier MA_1 represents the previously described aperiodic sections; MA_2 is an adder. In order to derive the transfer function we shall study the block diagram of the element (Fig. 6b). Here the following notation is used: k_1 is the current gain of MA_1 with respect to the control circuit; T is the time constant of the aperiodic section; k_3 is the current gain of MA_2 with respect to the control circuit; k_4 is the gain of MA_2 with respect to the negative feedback circuit; $k_d = R_{L2}/R_{C1}$ is the transfer coefficient of the divider (in order to eliminate the shunting effect it is desirable to make $k_d \ll 1$).

The transfer function of the element is written as

$$W(p) = \frac{k_3}{1 + \frac{k_1 k_4 k_d}{Tp + 1}} = \frac{k_3 (Tp + 1)}{Tp + 1 + k_1 k_4 k_d}. \tag{11}$$

We write the expression for the transfer function of the element in the following form when the effect of the capacitor leakage resistance in the amplifier MA_1 is taken into account:

---

* By increasing k_3 and reducing k_1 and k_4 it is possible to increase the integration time of the PI element without altering k_{st}.

$$W(p) = \frac{k_3\,[Tp + (1 + T/T_o)]}{Tp + (1 + T/T_{lk}) + k_1 k_4 k_d}. \tag{12}$$

The inequalities $T > \frac{T}{1 + k_1 k_4 k_d}$ and $\frac{T}{1 + T/T_{lk}} > \frac{T}{1 + T/T_{lk} + k_1 k_4 k_d}$ are always satisfied, and therefore the element can be used in controllers as a PD element.

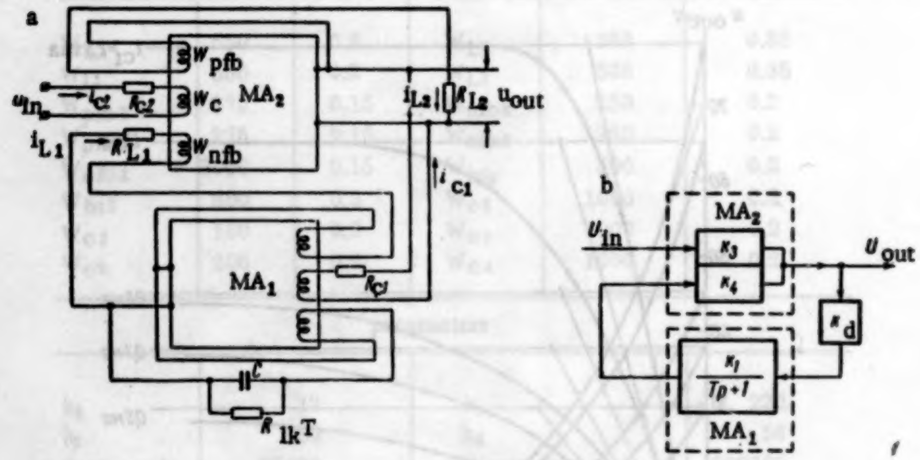


Fig. 6. a) Simplified circuit diagram; b) block diagram of the PD element.

We shall determine the parameters of this element from a study of the transient response when the current i_{c2} in the control winding of MA_2 varies according to a linear law (Fig. 7).

The current at the output of the PD element will be: when leakage of the capacitor in the RC circuit is neglected

$$i_{L2} = i_{c2}\frac{k_3}{1 + k_1 k_4 k_d}\left[t + \left(T - \frac{T}{1 + k_1 k_4 k_d}\right)\left(1 - \exp\left(-\frac{1 + k_1 k_4 k_d}{T}t\right)\right)\right], \tag{13a}$$

and when the leakage of the capacitor in the RC circuit is taken into account

$$i_{L2} = \frac{k_3(1 + T/T_{lk})}{1 + T/T_{lk} + k_1 k_4 k_d}\left[t + \left(\frac{T}{1 + T/T_{lk}} - \frac{T}{1 + T/T_{lk} + k_1 k_4 k_d}\right) \times \right.$$
$$\left. \times \left(1 - \exp\left(-\frac{1 + T/T_{lk} + k_1 k_4 k_d}{T}t\right)\right)\right] \tag{13b}$$

The proportionality coefficient $\underline{k}$ is equal to the ratio between the rates of change of the current in the load of the element and the control current in a steady-state mode (for $t \to \infty$):

$$k = \left[\frac{di_{L2}}{dt}\right]_{t\to\infty} \bigg/ \left[\frac{di_{c2}}{dt}\right]_{t\to\infty}. \tag{14}$$

The differentiation time constant T_d determines the time by which the load current i_{L2} leads the normalized current ki_{c2} in a steady-state mode. This time can be determined from the relationship (cf. Fig. 7)

$$i_{L2}(t + T_d) = k i_{c2}(t) \text{ for } t \to \infty. \tag{15}$$

TABLE 3

Type of PD element	Parameter of the PD element		
	k	T_d	k_q
With an ideal RC circuit	$\dfrac{k_s}{1+k_1k_4k_d}$	$T\dfrac{k_1k_4k_d}{1+k_1k_4k_d}$	$k_1k_4k_d$
With an RC circuit taking capacitor leakage into account	$\dfrac{k_s(1+T/T_{1k})}{1+T/T_{1k}+k_1k_4k_d}$	$T\dfrac{k_1k_4k_d}{(1+T/T_{1k})(1+T/T_{1k}+k_1k_4k_d)}$	$\dfrac{k_1k_4k_d}{1+T/T_{1k}}$

The coefficient k_q, which determines the quality of the differentiation, is equal to the ratio between the differentiation time constant and the transient response time constant [the time constant of the exponents in formulas (13)].

The formulas for determining the parameters of PD elements with different sections in the negative feedback circuit are given in Table 3.

From Table 3 it follows that all the parameters of magnetic amplifier PD elements depend on the transfer coefficient for the closed loop of MA_1 and MA_2; this coefficient is equal to the product $k_1k_4k_d$. In order to achieve

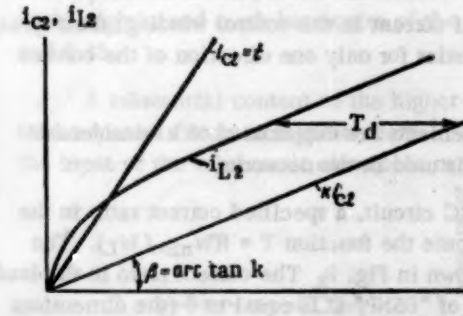


Fig. 7. Transient response in the PD element for a linearly increasing control current.

a better quality of differentiation one should choose $k_1k_4k_d \gg 1$. Under these conditions the differentiation time constant T_d for a PD element with an ideal RC circuit will be a maximum and equal to the time constant T of an aperiodic section; the value of T_d for a PD element with an RC circuit will differ from T by a factor $(1 + T/T_{1k})$ in this case when the leakage of the capacitor is taken into account. The proportionality coefficient in the PD elements under study depends both on the transfer coefficient of the loop and on the gain k_s with respect to the control circuit of the adder.

The parameters and operational stability of magnetic amplifier PD elements will increase as the RC circuits which are used in them approach the ideal (i.e., as the leakage in the capacitors decreases).

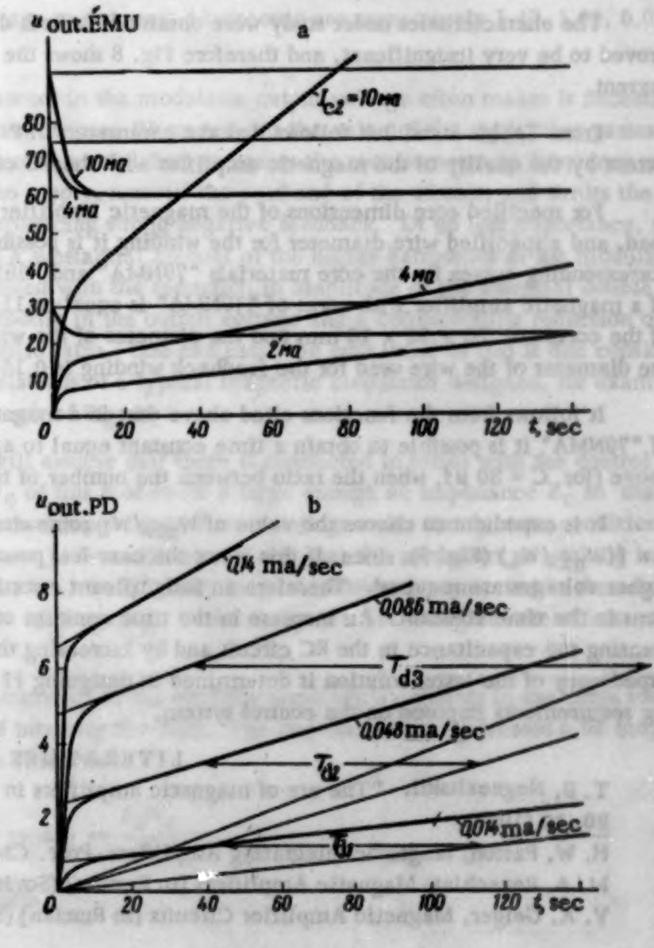


Fig. 8

The simplified circuit diagram of an experimental magnetic amplifier PD element based on the block diagra shown in Fig. 6 is given in Fig. 4. The parameters of MA_1 and MA_2 are cited in Table 2. The parameters of the PD element (the computation is performed in accordance with Table 3 for $T_{1k} \gg T$ and $k_d = 0.05$) are the following: the proportionality coefficient is $k = 5.9$ and less; the differentiation time constant is $T_d = 97.5$ sec and less; the coefficient for the quality of the differentiation is $k_q = 36.4$.

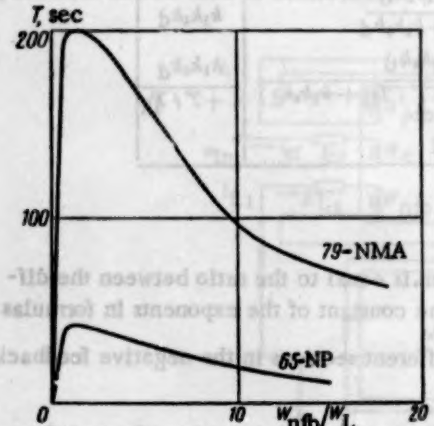


Fig. 9

The designed PD element (cf. circuit in Fig. 4) was used jointly with the simulator unit "ÉMU-8." We obtained the dynamic characteristics of a PD element with its output supplying the integrator of the "ÉMU-8"; here the input control current of the PD element was varied stepwise (Fig. 8a). We investigated the characteristics of the PD element when the input control current is varied according to a linear law at various rates (Fig. 8b). The differentiation time constants T_d in the linear range of operation of the PD element were equal to 80 sec in accordance with Fig. 8; i.e., they were approximately equal to the time constant of the magnetic amplifier with an RC circuit. In the nonlinear range of operation of the PD element the values of T_d increase. From the characteristics which are shown it is evident that the quality coefficient which characterizes the transient response in the PD element is close to 40. Thus the experimental parameters of the described PD element are in good agreement with the computed parameters (for $T_{1k} > 200$ sec).

The characteristics under study were obtained for both directions of current in the control winding; their spread proved to be very insignificant, and therefore Fig. 8 shows the characteristics for only one direction of the control current.

From Tables 1 and 3 it follows that the parameters of PI and PD elements are determined to a considerable extent by the quality of the magnetic amplifier with the RC circuit that is used in the networks.

For specified core dimensions of the magnetic amplifier with the RC circuit, a specified current ratio in the load, and a specified wire diameter for the winding it is possible to compute the function $T = f(W_{nfb} / W_L)$. The corresponding curves for the core materials "79NMA" and "65NP" are shown in Fig. 9. The current ratio in the load of a magnetic amplifier with cores of "79NMA" is equal to 11; for cores of "65NP" it is equal to 6 (the dimensions of the cores are $50 \times 32 \times 10$ mm and the diameter of the wire which is used for the operating winding is 0.2 mm; the diameter of the wire used for the feedback winding is 0.15 mm).

It follows from the functions cited above that in a magnetic amplifier with an RC circuit and toroidal cores of "79NMA" it is possible to obtain a time constant equal to approximately 200 seconds for the relationships cited above (for $C = 30\ \mu f$, when the ratio between the number of turns is $W_{nfb}/W_L = 3$).

It is expedient to choose the value of W_{nfb}/W_L somewhat to the right of the maximum on the curves for $T = f(W_{nfb}/W_L)$ (Fig. 9), since if this is not the case less power is drawn from the amplifier and supply sources with higher voltages are required. Therefore an insignificant error in choosing W_{nfb}/W_L can lead to appreciable reductions in the time constant. An increase in the time constant of PI and PD elements can be achieved both by increasing the capacitance in the RC circuit and by increasing the dimensions of the magnetic amplifier cores. The expediency of the latter solution is determined in designing PI and PD elements by taking into account the engineering requirements imposed on the control system.

LITERATURE CITED

1. T. B. Negnevitskii, "The use of magnetic amplifiers in integrating circuits," Avtomatika i Telemekhanika, 20, 10 (1959).
2. H. W. Patton, Magnetic Integrating Amplifiers. Proc. CMA, AIEE, Los Angeles (1958).
3. M. A. Rozenblat, Magnetic Amplifiers [in Russian] (Soviet Radio Press, 1960).
4. V. A. Geiger, Magnetic Amplifier Circuits [in Russian] (State Power Engineering Press, 1959).

MAGNETIC MODULATORS WITH A DOUBLE FREQUENCY SINUSOIDAL OUTPUT VOLTAGE

M. A. Rozenblat

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1386-1400, October, 1961
Original article submitted March 29, 1961

Conditions are determined for obtaining a sinusoidal output voltage shape from frequency doubling magnetic modulators which have magnetic excitation and signal fields that are either parallel or mutually perpendicular. It is shown that these modulators have the highest amplification and signal-to-noise ratio. A method is given for calculating such modulators.

1. Higher Harmonics, Coefficient of Transmission, and Lower Threshold of Sensitivity

Of all the known types of magnetic modulators those operating on the frequency doubling principle have potentially the lowest threshold of sensitivity. The voltage appearing at a modulator output when a signal is present contains, as a rule, not only the second harmonic of the supply frequency but also higher even harmonics. For example, in a typical series output modulator which is supplied with sinusoidal current the amplitude ratios of the fourth, sixth, eighth, and tenth harmonics of the output voltage to the second harmonic are respectively 1.47, 1.01, 0.62 and 0.43.

A substantial content of the higher even harmonics in the modulator output voltage often makes it necessary to use special filters to suppress them.* The presence of these filters reduces the value of the signal that passes to the input of the subsequent electronic or semiconductor ac amplifier, increases the requirements on frequency stability for the supply, narrows the passband of the circuit, and limits the possibility of introducing strong negative feedback. Of no less importance, the presence of a substantial amount of the higher harmonics at the modulator output is linked with the reduction in amplitude of the essential double frequency harmonic in the output voltage and a corresponding reduction of the signal-to-noise ratio. One can easily be convinced of this if one considers the basic relations of a typical magnetic modulator designed, for example, like the circuit of Fig. 1.

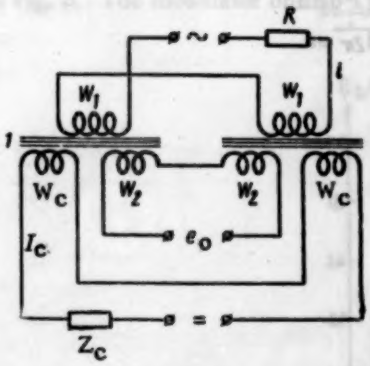


Fig. 1

We will assume that there is connected in series with the control windings W_c of this modulator a large enough ac impedance Z_c so that there is no ac component in these windings under steady state conditions (i.e., $i_c = I_c = \text{const}$). We will assume also that the modulator is working into an open circuit, i.e., an infinitely large resistance is connected to the terminals of its output windings W_2. In the analysis we will neglect the effect of hysteresis.

The current i in the exciting windings W_1 creates in the cores identical ac fields of intensity $H = H(t)$. The control current I_c creates a dc magnetic field in the first core:

$$\Delta H = H_c = \frac{I_c W_c}{l}, \tag{1}$$

and in the second core a similar field of opposite sign.

* When amplifying extremely weak signals it is often necessary to provide filters for the suppression of uneven harmonics due to imbalance, especially when using modulators with parallel fields.

The value of the voltage at the modulator output is found from the formula

$$e_0 = W_2 S \frac{d}{dt}(B_1 - B_2), \tag{2}$$

where S is the cross-sectional area of one of the cores, and B_1 and B_2 are the values of the magnetic induction in the respective modulator cores.

Confining ourselves to small signals it can be assumed that

$$B_1 = B(H + \Delta H) = B(H) + \frac{dB}{dH}\Delta H = B(H) + \mu_d \Delta H,$$

$$B_2 = B(H - \Delta H) = B(H) - \frac{dB}{dH}\Delta H = B(H) - \mu_d \Delta H,$$

where $\mu_d = dB/dH$ is the differential magnetic permeability.

Substituting in (2) the expressions obtained for B_1 and B_2 we find

$$e_0 = 2W_2 S \Delta H \frac{d\mu_d}{dt}. \tag{3}$$

The quantity μ_d has a maximum value (μ_{dmax}) when H = 0 and decreases monotonically with an increase in the absolute magnitude of H (Fig. 2a). If, for example, the excitation current i, and consequently the field intensity H, are varied according to a sinusoidal law ($H = H_m \sin \omega t$, Fig. 2b), then μ_d will vary at twice the frequency within the limits from μ_{dmin} to μ_{dmax} (Fig. 2c).

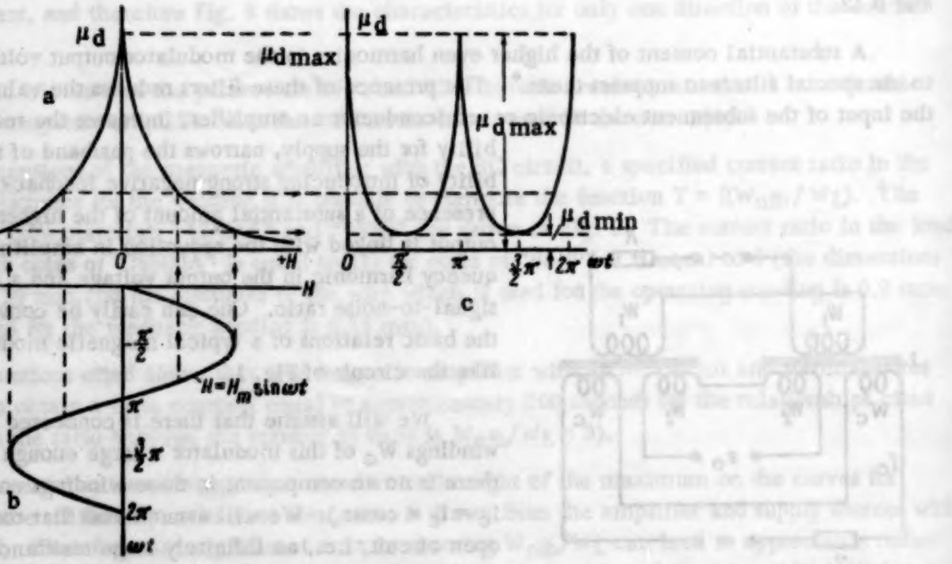


Fig. 2

The differential magnetic permeability is an even function of time $\mu_d(t) = \mu_d(-t)$ and can therefore be represented by means of the following series:

$$\mu_d = \sum_{n=0}^{\infty} \mu_{2n} \cos 2n\,\omega t.$$

For the case under consideration, corresponding to Fig. 2c, it is not difficult to show that $\mu_{2n} \geq 0$. This is the result of the monotonic character of the change in μ_d during a quarter-period.

When $\omega t = 0$

$$\mu_d(0) = \mu_{d\max} = \mu_0 + \mu_2 + \mu_4 + \mu_6 + \ldots, \tag{4}$$

and when $\omega t = \pi/2$

$$\mu_d\left(\frac{\pi}{2}\right) = \mu_{d\min} = \mu_0 - \mu_2 + \mu_4 - \mu_6 + \ldots \tag{5}$$

Consequently

$$\mu_{d\max} - \mu_{d\min} = 2(\mu_2 + \mu_6 + \mu_{10} + \ldots). \tag{6}$$

For a sufficiently large excitation current (or field) $\mu_{d\max} \gg \mu_{d\min}$ and

$$\mu_2 = \frac{\mu_{d\max}}{2} - (\mu_6 + \mu_{10} + \mu_{14} + \ldots). \tag{7}$$

We will now investigate the amplitude of the second harmonic of the output voltage, for which it is not difficult to find from (3) and (7)

$$E_2 = 4\omega W_2 S \Delta H \mu_2 = 4\omega W_2 S \Delta H \left[\frac{\mu_{d\max}}{2} - (\mu_6 + \mu_{10} + \mu_{14} + \ldots)\right]. \tag{8}$$

Obviously the maximum value of the second harmonic of the output voltage is equal to

$$E_{2\max} = 2\omega W_2 S \mu_{d\max} \Delta H, \tag{9}$$

which is achieved in the absence of the higher even harmonics of the output voltage. The relation is

$$\frac{E_2}{E_{2\max}} = \frac{2\mu_2}{\mu_{d\max}} = \frac{\mu_2}{\mu_2 + \mu_6 + \mu_{10} + \mu_{14} + \ldots}. \tag{10}$$

The relationships obtained are confirmed experimentally as indicated by the results of measurements presented in Fig. 3. The modulator of Fig. 1 was constructed on spiral toroidal cores made of 80NKhS alloy. The amplitudes

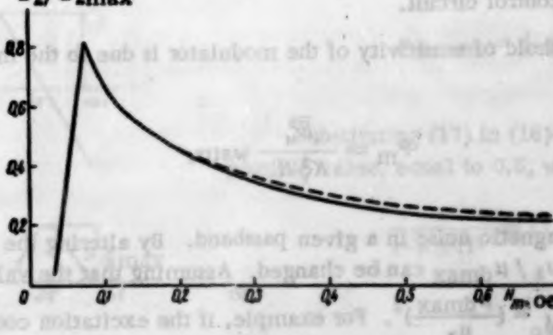


Fig. 3

of discrete harmonics in the output voltage (up to the fourteenth, inclusive) were measured with a voltage analyzer, which allowed the values of the corresponding components of the differential magnetic permeability to be calculated according to the formula

$$\mu_{2n} = \frac{E_{2n}}{2n\omega W_2 S \Delta H} = \frac{l E_{2n}}{2n\omega W_2 W_C S I_C}. \tag{11}$$

The magnitude of $E_{2\,max}$ is found by a measurement of the average value of modulator output voltage for which from (3) there can be obtained

$$E_{av} = \left| \frac{2}{\pi} \int_0^{\pi/2} e_0\, d\omega t \right| =$$

$$= \left| 8fW_2 S\Delta H \int_{\mu_{d\,max}}^{\mu_{d\,min}} d\mu_d \right| = 8fW_2 S\Delta H\,(\mu_{d\,max} - \mu_{d\,min}). \qquad (12)$$

For $\mu_{dmax} \gg \mu_{dmin}$ we find from (9) and (12)

$$E_{2\,max} = \frac{\pi}{2} E_{av}.$$

The solid curve in Fig. 3 depicts the dependence of the ratio $E_2/E_{2max} = 2\mu_2/\mu_{dmax}$ on the amplitude H_m of a sinusoidal exciting field for the modulator as measured by the indicated method. At first, the ratio E_2/E_{2max} grows with increasing H_m, which is due to the diminishing value of μ_{dmin}. When $H_m \approx 0.08$ oe, $\mu_{dmax} \gg \mu_{dmin}$, and further increase of H_m develops chiefly a greater content of the higher even harmonics in the curve $\mu_d = \mu_d(\omega t)$, at the expense of a smaller second harmonic μ_2.

The dashed curve in Fig. 3 is the function

$$\frac{\mu_2}{\mu_2 + \mu_6 + \mu_{10} + \mu_{14}} = f(H_m),$$

computed from measured values of μ_2, μ_6, μ_{10}, and μ_{14}.

As was to be expected, this function coming from formula (10) practically coincides with the dependence of E_2/E_{2max} on H_m. The small discrepancy which is seen at the relatively large values of H_m apparently is due still higher even harmonics which are beginning to develop at the indicated values of H_m.

The transmission coefficient (amplification) of the modulator is reduced in the same degree as the second harmonic of the output voltage is reduced (with respect to the maximum possible value E_{2max}):

$$K_V = \frac{E_2}{U_C} = \frac{E_2}{I_C R_C} = \frac{4\omega W_C W_2 S}{R_C l}\mu_2,$$

where R_C is the resistance of the control circuit.

The lower limit of the threshold of sensitivity of the modulator is due to the magnetic noise [1], expressed by the formula

$$P_m = \frac{\overline{E}_M^2}{K_V^2 R_C} \text{ watts},$$

where $\overline{E}_M$ is the voltage of the magnetic noise in a given passband. By altering the shape of the exciting current curve of the modulator the ratio μ_2/μ_{dmax} can be changed. Assuming that the value of $\overline{E}_M$, moreover, is practically unchanged, we find that $P_M \equiv \left(\frac{\mu_{dmax}}{\mu_2}\right)^2$. For example, if the excitation conditions of the modulator are such that μ_2 is four times smaller than the maximum possible value of $0.5\,\mu_{dmax}$, then the lower threshold of the modulator sensitivity becomes 16 times higher than the value attained for a sinusoidal shape of the output voltage.

It should be noted that such an output voltage shape is attained with a more uniform time rate of change in the magnetic induction of the cores than when the modulator is excited by means of a sinusoidal current or voltage (see below). Therefore the magnetizing increments (Barkhausen increments) also will be more uniformly distributed, which is apparently accompanied by a reduction in the absolute value of $\overline{E}_M$. Considering also that a sinusoidal output voltage shape from the modulator often eliminates the need for an output filter together with the losses in it, in such cases a still more substantial increase in signal-to-noise ratio can be expected.

2. Conditions for Obtaining a Sinusoidal Output Voltage Shape in Modulators with Parallel Fields

Modulators with cores having magnetic exciting and signal fields in parallel (or antiparallel) directions are arranged in accord with the circuits presented in Fig. 1. For such modulators a sinusoidal shape of the output voltage can obviously be obtained if

$$\mu_d = \mu_0 + \mu_2 \cos 2\omega t. \tag{13}$$

Considering that $\mu_{dmax} = \mu_0 + \mu_2$, it is possible to write

$$\mu_d = \mu_{d\,max}(1 - \xi + \xi \cos 2\omega t), \tag{14}$$

where

$$\xi = \frac{\mu_2}{\mu_{d\,max}} = \frac{\mu_{d\,max} - \mu_{d\,min}}{2\mu_{d\,max}} \leqslant 0.5. \tag{15}$$

If the magnetizing curve of the modulator cores is given analytically in the form of a function B = f(H), then by solving the equation

$$f'(H) = \mu_{d\,max}(1 - \xi + \xi \cos 2\omega t), \tag{16}$$

it is possible to find the law of the time rate of change of the exciting field intensity H or the magnetic induction B for which the modulator output voltage will be sinusoidal.

Let us look at some examples.

It is known that the average curve of magnetization (that is, the average between the rising and descending branches of the limiting hysteresis loop) for a number of materials used in modulators (for example, 80NKhS alloy, ferrocart-2000, and others) can be approximated with sufficient accuracy by the arctangent [1]

$$B = \frac{2}{\pi} B_s \operatorname{arc\,tg} \beta H = f(H). \tag{17}$$

From this expression it can be seen that when $H \to \infty$, the induction $B \to B_s$, where B_s is the induction for saturation. Moreover,

$$\mu_{d\,max} = \lim \left(\frac{dB}{dH}\right)_{H \to 0} = \frac{2}{\pi} \beta B_s. \tag{18}$$

Substituting (17) in (16) and assuming for ξ the maximum possible value, equal to 0.5, we find, taking into account (18),

$$\frac{1}{1 + \beta^2 H^2} = (1 + \cos 2\omega t) = \cos^2 \omega t$$

or

$$\beta H = \pm \operatorname{tg} \omega t.$$

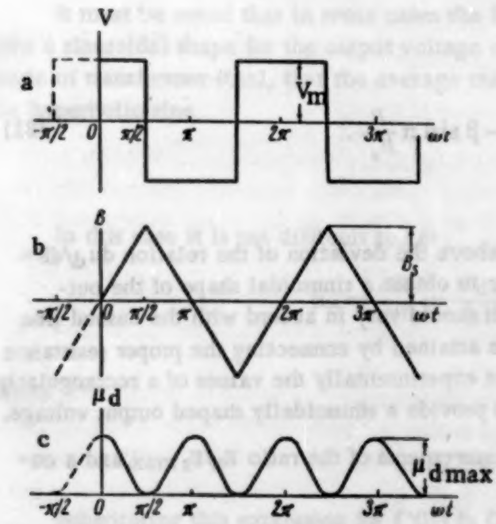


Fig. 4

Substituting this value in (17) we obtain the required law for the change of magnetic induction:

$$B = \pm \frac{2}{\pi} B_s \omega t = \pm 4fB_s t. \tag{19}$$

Such a law for the change in magnetic induction is obtained when the modulator is supplied from a source of alternating voltage with rectangular shape and an amplitude of

$$V_m = 8fW_1SB, \tag{20}$$

provided that the voltage drop in the resistance of the exciting circuit can be neglected. In Fig. 4 there are shown diagrams of the change in supply voltage, in magnetic induction, and magnetic permeability for the case being considered.

If the magnetization curve of the cores cannot be approximated with the arctangent, or if it is impossible to neglect completely the voltage drop in the resistance of the excitation circuit (for example, when $B \to B_s$), then the shape of the modulator output voltage will not be purely sinusoidal. However, the observed deviations in the magnetization curves for cores made of the magnetic materials most often used for magnetic modulators (for example, alloys 80NKhS, 79NM, 79NMA, ferrocart-2000), as compared with the magnetization curve corresponding to the arctangent (17), generally produce distortions in the shape of the modulator output voltage which can be easily compensated by a proper selection of the resistance R in the excitation circuit. By considering the curves in Fig. 5 one can easily verify this.

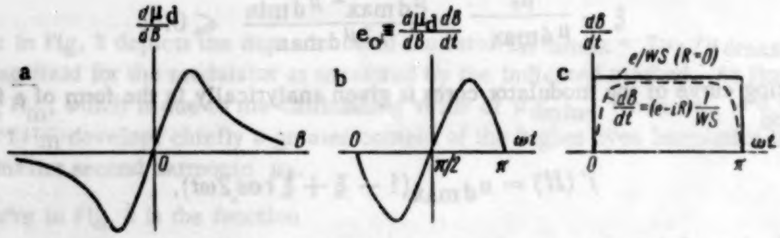


Fig. 5

The modulator output voltage found from formula (3) can be also written in the following form:

$$e_0 = 2W_2 S \Delta H \frac{d\mu_d}{dB}\frac{dB}{dt}.$$

In the example considered above $dB/dt = \pm 4fB_s$, and

$$\frac{d\mu_d}{dB} = \frac{d}{dB}\left(\frac{\frac{2\beta B_s}{\pi}}{1+\mathrm{tg}^2\frac{\pi}{2}\frac{B}{B_s}}\right) = -\beta \sin \pi \frac{B}{B_s}. \tag{21}$$

For toroidal cores made from the magnetic materials specified above the deviation of the relation $d\mu_d/dB = f(B)$ from the sinusoidal (21) is of the type shown in Fig. 5a. In order to obtain a sinusoidal shape of the output voltage $e_0 \equiv (d\mu_d/dB)\cdot(dB/dt)$ (Fig. 5b), it is necessary that dB/dt should vary in accord with the dashed line in Fig. 5c. The variation $dB/dt = f(\omega t)$ close to that required can be attained by connecting the proper resistance into the excitation circuit. In practice it is extremely simple to select experimentally the values of a rectangularly shaped supply voltage and a resistance for the excitation circuit which provide a sinusoidally shaped output voltage.

As an example there are presented in the table the results of measurements of the ratio $E_2/E_{2\,max}$ and a coefficient of nonlinear distortion for the output voltage of a modulator

$$K_D = \frac{\sqrt{E_4^2 + E_6^2 + E_8^2 + E_{10}^2 + E_{12}^2 + E_{14}^2}}{E_2}$$

made with cores of 79NMA alloy, for various excitation conditions. For each case here the same average value of the ac excitation field intensity was established, equal to 0.1 oe.

Note that when the field intensity of excitation is increased, the shape of the output voltage deteriorates sharply for a modulator excited by sinusoidal current or sinusoidal voltage, but when excited by rectangular-shaped voltage, a curve of relatively low distortion can be obtained by a suitable change in the value of a series-connected resistance.

Method of excitation	Sinusoidal current	Sinusoidal voltage	Voltage of rectangular shape
E_2/E_{2max}	0.56	0.68	0.93
K_D	1.43	0.97	0.2

In a modulator having cores of 79NM alloy which was supplied with rectangularly shaped voltage,still less distortion was obtained in the shape of the output voltage: $K_D = 0.06$ for $E_2/E_{2max} = 0.97$.

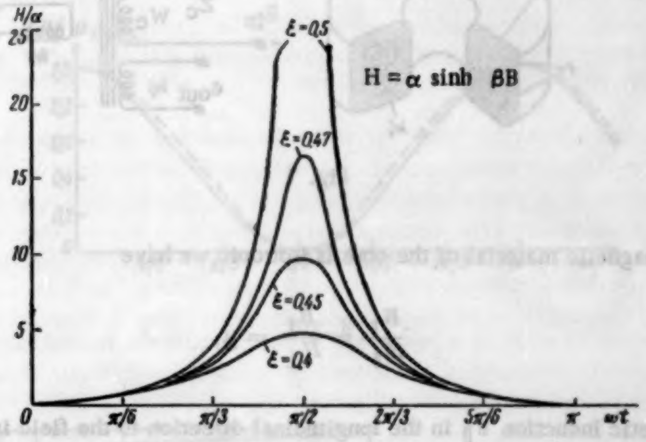


Fig. 6

It must be noted that in some cases the fixing of a certain shape of the excitation current curve which will give a sinusoidal shape for the output voltage can be done as expedient. Let us assume, as often occurs for cores made of transformer steel, that the average magnetization curve can be approximated with sufficient accuracy by the hyperbolic sine

$$H = \alpha \operatorname{sh} \beta B.$$

In this case it is not difficult to get

$$\mu_d = \frac{dB}{dH} = \frac{\mu_{d\,max}}{\cosh \beta B} = \frac{\mu_{d\,max}}{\sqrt{1+\operatorname{sh}^2\beta B}} = \frac{\mu_{d\,max}}{\sqrt{1+\left(\frac{H}{\alpha}\right)^2}},$$

where

$$\mu_{d\,max} = \frac{1}{\alpha\beta}.$$

Substituting this expression for f'(H) in (16) and solving for H we find

$$\frac{H}{\alpha} = \pm\sqrt{\frac{4\xi \sin^2 \omega t\,(1-\xi \sin^2 \omega t)}{1-4\xi \sin^2 \omega t\,(1-\xi \sin^2 \omega t)}}. \tag{22}$$

In Fig. 6 there are presented graphs of the required shape for the excitation field (or current) intensity(in one half-period),which were plotted from formula (22) for various values of ξ. When $\xi \approx 0.4$, the required shape of the excitation current is attained with sufficient accuracy if the modulator is supplied from a source of rectangularly shaped voltage. For higher values of ξ it is necessary to take supplementary measures in order to get the required shape of the excitation current. A satisfactory method of changing the shape of the excitation current within broad limits is to connect a saturating choke coil in the excitation circuit of the modulator.

3. Magnetic Modulators with Mutually Perpendicular Fields

The design and the basic circuit of the modulator are shown in Fig. 7. A ferrite core 1 is composed of two identical halves having ring-shaped grooves in which the excitation winding W_1 is located. The control winding W_c and the output winding W_2 are arranged uniformly over the entire length of the core.

In this modulator the output voltage can no longer be calculated from formula (3) because the increase in the magnetic field $\Delta H = H_c$ due to the input signal does not add algebraically to the excitation field H.

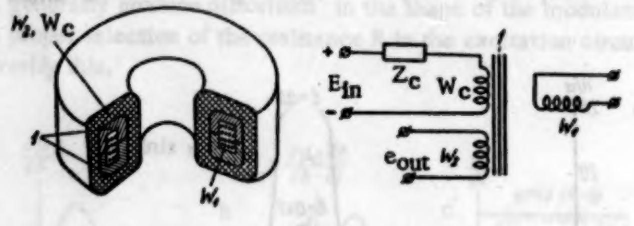


Fig. 7

Assuming that the magnetic material of the core is isotropic we have

$$\frac{B_{\parallel}}{H_{\parallel}} = \frac{B_{\perp}}{H_{\perp}} = \mu, \tag{23}$$

i.e., the ratio of the magnetic induction $B_{\parallel}$ in the longitudinal direction to the field intensity $H_{\parallel}$ acting in this direction is equal to the ratio of the induction $B_{\perp}$ in the transverse direction to the field intensity $H_{\perp}$ acting in the transverse direction. Since $H_{\parallel} = \Delta H = H$ is the signal field intensity, then for the output voltage of the modulator we have

$$e_0 = W_2 S_{\parallel} \frac{dB_{\parallel}}{dt} = W_2 S_{\parallel} \Delta H \frac{d\mu}{dt} = W_2 S_{\parallel} \Delta H \frac{d\mu}{dB} \frac{dB}{dt}, \tag{24}$$

where $S_{\parallel}$ is the cross-section area of the core for the longitudinal magnetic field.

The principal difference between this formula and formula (3) for the modulator with parallel fields is that in expression (24) for the modulator with mutually perpendicular fields μ is the static magnetic permeability of the core while μ_d in (3) is the differential magnetic permeability. Moreover, both μ_d and μ are determined by the average magnetization curve, which does not have points of discontinuity. Therefore

$$\mu_{max} = \left(\frac{B}{H}\right)_{B\to 0} = \left(\frac{dB}{dH}\right)_{B\to 0} = \mu_{d\,max}.$$

Confining ourselves to small signals it can be assumed that the function $\mu(t)$ is identically determined by the law of change of the transverse induction component $B_{\perp}$ (or $H_{\perp}$) and does not depend on the value of the signal. Therefore instead of equation (16) it is possible to write the following condition for the sinusoidal voltage at the output of a modulator having mutually perpendicular fields:

$$\frac{B_{\perp}}{H_{\perp}} = \mu = \mu_{max}(1 - \xi + \xi \cos 2\omega t). \tag{25}$$

If the magnetization curve is approximated by the arctangent (17), then a sinusoidal output voltage will be obtained for different excitation conditions than for the case of the modulator with parallel fields.

Actually, for the case under consideration it is possible in practice to get a sinusoidal output voltage shape by varying the induction according to a sinusoidal law

$$B_{\perp} = B_s \sin \omega t. \tag{26}$$

From (17) and (26) we find for the change of magnetic permeability with time

$$\frac{\mu}{\mu_{\max}} = \frac{1}{\mu_{\max}} \frac{B_{\perp}}{H_{\perp}} = \frac{\frac{\pi}{2}\sin \omega t}{\operatorname{tg} \frac{\pi}{2} \sin \omega t}. \tag{27}$$

The solid curve in Fig. 8 represents the required law for the change of μ when $\xi = 0.5$, as established by the right-hand part of equation (25). The dashed curve was computed from formula (27) and corresponds to the change of induction in accord with (26).

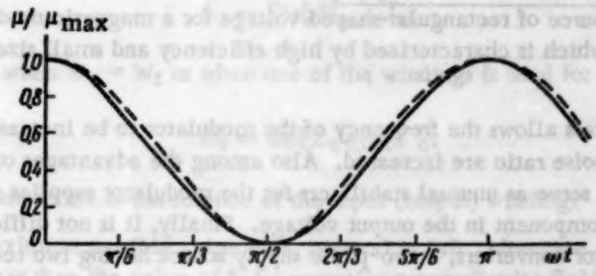


Fig. 8

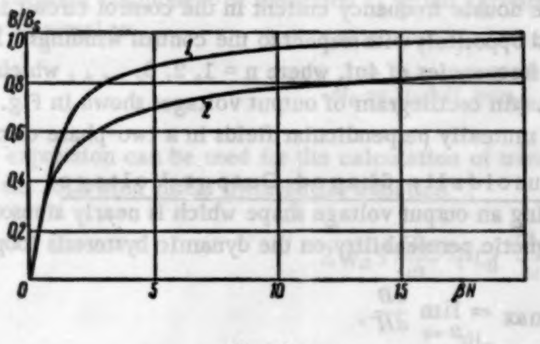


Fig. 9

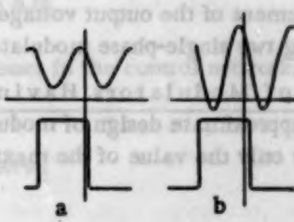


Fig. 10

In Fig. 9 curve 1 represents the function B = f(H) which is described by the arctangent (17). It is interesting to note that if the magnetization curve can be approximated by the function

$$H = \frac{1}{\beta} \frac{\pi \frac{B}{B_s}}{1 + \cos \pi \frac{B}{B_s}}, \tag{28}$$

which corresponds to curve 2 in Fig. 9, then the modulator output voltage will be sinusoidal when the change of induction is directly proportional to time, in accord with (19), i.e., when the modulator excitation is from a source of ac voltage of rectangular shape. Indeed, substituting (19) in (28) we find

$$\mu = \frac{B}{H} = \frac{\beta B_s}{\pi}(1 + \cos 2\omega t) = 0.5\mu_{\max}(1 + \cos 2\omega t).$$

From Fig. 9 it is seen that magnetization curve 2 found from formula (28) coincides in the area of low induction with magnetization curve 1 which corresponds to the arctangent (17), but in the area of high induction it lies below the other. This type of magnetization curve is observed for cores of the kind represented in Fig. 7 as a result of non-uniform core magnetization by the transverse magnetic excitation field. The variation of magnetic path

length over wide limits and also the presence of a varying area of core cross section for the transverse field lead to incomplete saturation, i.e., at a time when part of the core is saturated, another part proves to be unsaturated. Therefore the curve representing the ratio of the average value of magnetic induction over the cross section of the core will lie below the magnetization curve of the core material in the area of high induction.

An experimental study of a large number of modulators having mutually perpendicular fields which were made with cores of ferrocart-2000 demonstrated that it is possible to get in practice a sinusoidal output voltage shape by an appropriate choice of the amplitude of a rectangular-shaped supply voltage and of the resistance in the excitation circuit [2]. In Fig. 10a, there is presented a typical oscillogram of the supply and output voltages obtained for such modulators.

4. Modulators with a Two-Phase Supply

The most convenient source of rectangular-shaped voltage for a magnetic modulator supply is a magnetic-transistor ac to dc converter which is characterized by high efficiency and small size (see, for example, [1], ch. 7 to 15).

The use of such converters allows the frequency of the modulator to be increased, and thus its coefficient of amplification and signal-to-noise ratio are increased. Also among the advantages of magnetic-semiconductor converters are the facts that they serve as unusual stabilizers for the modulator supplies and they can be built with a negligible second harmonic component in the output voltage. Finally, it is not difficult to construct, on the principle of the magnetic-transistor converters, a two-phase supply source having two rectangularly shaped output voltages shifted in phase by 90° [2].

The use of two modulators supplied by a two-phase source permits the second harmonic to be eliminated in the control circuit formed by the series connection of the modulator control windings. Therefore the need of inserting a filter or an impedance which is rather large for the double frequency current in the control circuit is eliminated. The output windings of the modulators are connected oppositely with respect to the control windings. In addition, at the modulator output there will be no voltages with frequencies of 4nf, where n = 1, 2, 3, . . . , which provides a further improvement of the output voltage shape [2]. An oscillogram of output voltages shown in Fig. 10b was taken after combining two single-phase modulators having mutually perpendicular fields in a two-phase circuit.

5. Design of Modulators Having a Sinusoidally Shaped Output Voltage

For the approximate design of modulators having an output voltage shape which is nearly sinusoidal, it is sufficient to know only the value of the maximum magnetic permeability on the dynamic hysteresis loop:

$$\mu_{d\,\max} = \lim_{B \to 0} \frac{dB}{dH}.$$

This quantity determines the value of the maximum permeability on the average magnetization curve

$$\mu_{\max} = \lim_{B \to 0} \frac{B}{H} = \mu_{d\,\max}.$$

Substituting (17) in (3) we find for the amplitude of the second harmonic at the output of a modulator having parallel fields, when it is operating in the open circuit condition (infinitely large load resistance)

$$E_{2\parallel} = 4\omega\xi W_2 S \mu_{d\,\max} \Delta H = \frac{2\omega\xi 2 W_2 W_c S \mu_{d\,\max}}{l} I_c, \qquad (29)$$

where

$$\Delta H = \frac{W_c I_c}{l}.$$

For a modulator having mutually perpendicular fields from (24) and (25) we have

$$E_{2\perp} = 2\omega\xi \frac{W_2 W_c S_\parallel \mu_{\max}}{l} I_c. \qquad (30)$$

We notice that

$$\frac{2W_2 W_c S \mu_{d\,max}}{l} = M_{max}$$

and

$$\frac{W_2 W_c S_{\parallel} \mu_{max}}{l} = \mu_{max}.$$

where M_{max} is the maximum value of the coefficient of mutual inductance between modulator windings W_c and W_2. Therefore for both the the modulator with parallel fields and the modulator with mutually perpendicular fields

$$E_2 = 2\omega\xi M_{max} I_c. \tag{31}$$

In the particular case when $W_c = W_2$ or when one of the windings is used for an input or an output we get

$$E_2 = 2\omega\xi L_{c\,max} I_c, \tag{32}$$

where $L_{c\,max}$ is the maximum value of inductance of the input (output) winding.

We recall that the maximum possible value of ξ is equal to 0.5. Moreover, the value of ξ in the case of parallel fields is always larger than the value of ξ for mutually perpendicular fields. In the first case it is possible to take $\xi = 0.5$ without a substantial error, and in the second case, as experimental studies and calculations showed, $\xi \approx 0.4$ to 0.45.

The constant component of induction B_0 which appears in the core due to the signal is calculated from (16) and (25) to be equal to

$$B_0 = \mu_0 \Delta H = \mu_{max}(1-\xi)\Delta H. \tag{33}$$

This expression can be used for the calculation of transient processes in the control network, which are described by the following linear differential equation* :

$$2W_c S \frac{dB_0}{dt} + L_d \frac{di_c}{dt} + R_c i_c = v_c$$

or

$$(L_c + L_d)\frac{di_c}{dt} + R_c i_c = v_c,$$

where L_d is the inductance of the choke connected in series with the control winding and L_c is the inductance of this winding for input signals, equal in the case of parallel fields to

$$L_c = \frac{2W_c^2 S(1-\xi)\mu_{d\,max}}{l} = \frac{W_c}{W_2}(1-\xi)M_{max}$$

and in the case of mutually perpendicular fields to

$$L_c = \frac{W_c^2 S_{\parallel}(1-\xi)\mu_{max}}{l} = \frac{W_c}{W_2}(1-\xi)M_{max}.$$

When a resistance R_L is present in the output of the modulator, a current i_L flows through its winding W_2 creating a field intensity H_L which has an effect on the value of the emf induced in the output winding of the modulator. The value of the load current is found from the differential equation of the load circuit

* This equation is written for a modulator with parallel fields and two cores. For a modulator with perpendicular fields it is necessary to substitute $2S = S_{\parallel}$.

$$2W_2 S \frac{d\mu_d \Delta H}{dt} + R_L i_L = 0. \quad (34)$$

For simplicity we will only consider the effect of the second harmonic of the current i_L. Then

$$i_L = I_2 \sin(2\omega t - \varphi).$$

This current creates a field intensity

$$H_L = \frac{i_L W_2}{l} = \frac{I_2 W_2}{l} \sin(2\omega t - \varphi) = H_2 \sin(2\omega t - \varphi),$$

which adds to the signal field $H_c = I_c W_c / l$. Therefore we have for the value of ΔH in formula (34)

$$\Delta H = H_c + H_2 \sin(2\omega t - \varphi).$$

Assuming as before that the time rate of change of μ_d is determined by the supply voltage and is described by formula (14), and considering only the second harmonic, we find from (34), after substitutions for the values of μ_d, ΔH, and i_L,

$$2\omega L_i \left[-\frac{\xi}{1-\xi} H_c \sin 2\omega t + H_2 \cos(2\omega t - \varphi)\right] + R_L H_2 \sin(2\omega t - \varphi) = 0, \quad (35)$$

where

$$L_i = \frac{2W_2^2 S (1-\xi)\mu_{d\max}}{l} = \frac{W_2^2}{W_c^2} L_c = \frac{W_2}{W_c}(1-\xi) M_{\max} \quad (36)$$

is the average value of the inductance of the modulator output winding.

Assuming in (35) that $\omega t = 0$ we find

$$\operatorname{tg} \varphi = \frac{2\omega L_i}{R_L}.$$

Substituting in (35) the value $2\omega t = \pi/2$, we obtain

$$H_2 = \frac{2\omega L_i \frac{\xi}{1-\xi} H_c}{\sqrt{R_L^2 + (2\omega L_i)^2}}. \quad (37)$$

Now it is possible to find the amplitude of the second harmonic voltage at the output of a modulator with an impedance load:

$$V_2 = I_2 R_L = \frac{H_2 l}{W_2} R_L = \frac{2 R_L \omega L_i \frac{\xi}{1-\xi}}{\sqrt{R_L^2 + (2\omega L_i)^2}} \frac{W_c}{W_2} I_c. \quad (38)$$

Expressing L_i in the numerator of (38) in terms of $M_{\max}$, according to (36), and considering (31), we find

$$V_2 = \frac{E_2 R_L}{\sqrt{R_L^2 + (2\omega L_i)^2}}. \quad (39)$$

This formula is the theoretical basis of the empirical rule established earlier for the approximate output voltage calculation of a double frequency modulator, in which the modulator is considered as a generator of an emf E_2 which has an internal inductive reactance (see [1], ch. 9, sec. 5C, and [2]):

$$Z_l = 2\omega L_l = 2\omega L_C \frac{W_2^2}{W_C^2}.$$

It is obvious that by this method it is also possible to calculate approximately the effect of a load having an inductive or capacitative character.

With an operating load the maximum value of output power, which is achieved when $R_L = 2\omega L_l$, is found from (39), taking into consideration (36):

$$P_L = \frac{V_2^2}{2R_L} = \frac{E_2^2}{8\omega L_l} = 0.5\left(\frac{\xi}{1-\xi}\right)^2 \omega L_C I_C^2. \qquad (40)$$

For the power amplification coefficient we have

$$K_p = \frac{P_L}{P_C} = \frac{P_L}{I_C^2 R_C} = 0.5\left(\frac{\xi}{1-\xi}\right)^2 \frac{\omega L_C}{R_C}, \qquad (41)$$

where R_C is the resistance of the control circuit. If the modulator control circuit includes, besides an inductance L_C in the winding W_C, only a resistance R_C then the ratio $L_C/R_C = \tau_C$ is the time constant of the control circuit and

$$\frac{K_p}{\tau_C} = \pi\left(\frac{\xi}{1-\xi}\right)^2 f. \qquad (42)$$

For a two-phase magnetic modulator the values of E_2, L_l, and R_C are doubled. Therefore P_L is also doubled with respect to the value given by formula (40) while K_p, τ_C, and K_p/τ_C keep the same values as for a single-phase modulator.

SUMMARY

The studies which have been made show a substantial possibility of achieving by simple means in practice a sinusoidally shaped voltage at the output of modulators operating on the principle of frequency doubling. To this end it is satisfactory in the majority of cases to supply a modulator from a source of rectangularly shaped voltage. In addition to this the amplification coefficient of the modulator increases and the signal-to-noise ratio is improved.

For an approximate calculation of the static and dynamic characteristics of modulators having a sinusoidally shaped output voltage it is sufficient to know only the maximum value of magnetic permeability on the dynamic hysteresis loop of the core material.

The conditions for producing a sinusoidally shaped output voltage for modulators with parallel and with mutually perpendicular fields differ in principle. However, for the same cores and winding data these regulators have practically the same static and dynamic characteristics.

When two of the modulators having mutually perpendicular fields are supplied from a two-phase source of rectangularly shaped voltage, it is possible to set up a magnetic modulator which has an extremely low threshold of sensitivity without filters in the supply, signal, or output circuits [2].

LITERATURE CITED

1. M. A. Rozenblat, Magnetic Amplifiers [in Russian], Izd. Sovetskoe Radio (1960).
2. F. I. Kerbnikov and M. A. Rozenblat, "Sensitive magnetic modulator with two-phase input," Avtomatika i Telemekhanika, 22, No. 3 (1961).

THE EFFICIENCY OF INFORMATION TRANSMISSION IN TELEMETERING

II. A POINT OF DEPARTURE FOR AN ANALYSIS TAKING ACCOUNT OF INTERFERENCE

N. V. Pozin

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1401-1404, October, 1961
Original article submitted April 13, 1961

Expressions are derived for the transmission rate of telemetered data, taking into account both equipment errors and the effect of interference in the communication channel.

The concepts of resultant precision and the criterion of interference stability of telemetering are presented.

In the first portion of article [1], in the analysis of the efficiency of information transmission without taking account of interference in the communication channel, it was considered that the telemetering precision N_A is determined only by the equipment error of the conversion transducer used (which was also considered identical to the discreteness error), that is, it was considered that

$$N_A = \frac{1}{\delta_A}, \tag{1}$$

where δ_A is the equipment error and N_A is the equipment precision of the telemetering.

In the general case, taking into account the effect of interference arising in the communications channel, we can say that the information received reproduces the value of the telemetered parameter with a resultant error δ, determined both by the equipment error δ_A and by the error δ_i caused by interference, which are, in general, quantities independent of each other. Before defining δ we call attention to the fact that the equipment error δ_A is usually taken as a maximum. It represents the maximum of the absolute deviation (averaged over the set of values at a given point of the scale), referred to the range of measurement. The error δ_i caused by interference is defined as a mean-square error or a mean error (see [2-4]).

We define the resultant error δ as a maximum, similar in physical meaning to the equipment error.

If δ_i is a mean-square error, the square of which is proportional to the dispersion of the distribution of errors caused by interference, then by the rule of summation of the dispersions of independent quantities we obtain

$$\left(\frac{\delta}{r}\right)^2 = \left(\frac{\delta_A}{r_A}\right)^2 + \delta_i^2$$

or

$$\delta = \sqrt{\left(\frac{r}{r_A}\right)^2 \delta_A^2 + r^2\delta_i^2}, \tag{2}$$

where r and r_A are the proportionality factors between the maximum value and the mean-square value of the resultant error and the equipment error respectively.*

* Obviously, if δ (or δ_A) obeys the normal distribution law, then the corresponding r (or r_A) may be set equal to 3.

If δ_i is a mean error, then, adding the mean values of the errors, we obtain the following expression for the resultant error:

$$\delta = \frac{r'}{r'_A}\delta_A + r'\delta_i,$$

where r' and r'_A are the proportionality factors between the maximum and mean values of the resultant and equipment errors respectively.*

We now define N as the resultant precision or resultant number of quantizing levels into which we subdivide the scale of measurements in reception as a result of the effect of interference and instability of equipment:

$$N = 1/\delta. \tag{3}$$

We shall now proceed to define the rate of information transmission in telemetering.

For a uniform distribution of the measured parameter (information) and for the new interpretation of the telemetering precision N, the information transmission rate R in the presence of interference in the communication channel is expressible by the same well-known general formula

$$R = \frac{\log N}{T}, \tag{4}$$

which was used in the case with no interference [1]. The difference here consists in the fact that instead of the equipment error the formula contains the resultant precision N. In this formula T is the transmission time of one measurement.

Let us consider the value of log N. According to (3)

$$\log N = -\log\delta.$$

Subdividing this into two terms, of which the first does not depend on interference, we obtain

$$\log N = -\log\delta_A - \log\frac{\delta}{\delta_A}.$$

We introduce the notation

$$Q = \frac{\delta_A}{\delta}. \tag{5}$$

We shall call Q the criterion of interference-stability of telemetering transmission. The fact is that the error δ_i caused by interference is not yet an exhaustive criterion of interference-stability. Such a criterion must be based on a comparison of the equipment error δ_A with the resultant error δ or with the error caused by interference δ_i.** In particular, the ratio

$$q_1 = \frac{\delta_A}{\delta_i} \tag{5a}$$

may also serve as a criterion of interference-stability. However, the criterion Q($1 > Q > 0$ when $0 < \delta_i < \infty$) is preferable in the present case, since it leads to simpler expressions for the transmission rate.

Taking (1) into account, we obtain

$$\log N = \log N_A - \log\frac{1}{Q} = \log N_A Q.$$

* The factor r' (or r'_A) is defined by the nature of the functional dependence of the absolute deviations on the value of the measured parameter. For example, for a linear dependence we have a factor r' (or r'_A) equal to 2.

** We note that an analysis and comparison of interference-stability of telemetering methods should also be carried out using as a basis a criterion of the type of the proposed interference-stability criterion Q, whose value depends on the relative values of the errors δ_A and δ (or δ_A and δ_i).

We make use of the expression (4). The rate of transmission of information in telemetering in the presence of interference in the communication channel is equal to

$$R = \frac{\log N_A}{T} - \frac{\log \frac{1}{Q}}{T}. \tag{6}$$

In accordance with the structure of this formula – two terms on the right side – we can represent the transmission rate as consisting of two parts:

$$R = R_A - \Delta R_1, \tag{7}$$

where R_A is the transmission rate in the case when there is no interference, a rate determined by the equipment error (it was just this value of the transmission rate that was considered in [1]). ΔR_1 is the loss in transmission rate caused by interference in the communication channel.

From (2), (5), and (6) it is clear that if the error δ_1 caused by interference becomes zero, then there will be no loss in rate ($\Delta R_1 = 0$), since the interference-stability Q becomes equal to unity.

Let us consider another definition of the transmission rate of telemetered information, suitable in a number of cases for the analysis of methods of telemetering transmission (see [1]). The difference in this definition consists in the fact that the rate of transmission of telemetered data (which in the present case we designate by V) is expressed not by logarithmic units per second but by the number of quantizing levels per second

$$V = N/T. \tag{8}$$

The resultant precision N may be represented as consisting of two parts:

$$N = N_A - \Delta N_1, \tag{9}$$

where ΔN_1 is the loss in precision caused by interference.

Making use of (1) and (3), it is easy to convince ourselves that

$$\Delta N = N_A (1 - Q),\ N = N_A Q.$$

On the basis of (9) we represent the rate V also as consisting of two parts:

$$V = V_A - \Delta V_1. \tag{10}$$

Here $V_A = N_A/T$; the second term $\Delta V_1 = \Delta N/T$ is the loss in transmission rate caused by interference.

In [1] we obtained expressions for the transmission time of one measurement T for the basic methods of transmission of information in telemetering. In [2-4] formulas were derived for the errors caused by interference in transmission under conditions of weak and strong interference.

These data are sufficient to obtain on the basis of (7) and (10) expressions for the transmission rate R or V, as well as the transmission efficiency criterion R/W considered in [1] (where W is the bandwidth occupied in the communication channel) for each method of modulation.

The values of R and R/W (or V and V/W) for concrete telemetering systems make it possible to estimate and compare the transmission efficiencies of different systems. However, since the expressions R and R/W (or V and V/W), obtained in this manner are functions of the basic parameters characteristic of the telemetering, it is of interest to carry out an analysis of these expressions for the purpose of finding the conditions of most efficient transmission.

LITERATURE CITED

1. N. V. Pozin, "The efficiency of information transmission in telemetering: I. Analysis without considering interference in the communication channel," Avtomatika i Telemekhanika, 22, No. 9 (1961).
2. V. A. Kashirin and G. A. Shastova, "The interference-stability of the transmission of telemetered signals in a channel with fluctuation interference," Avtomatika i Telemekhanika, 19, No. 8 (1958).
3. N. V. Pozin, "On the interference-stability of pulse-frequency telemetering," Avtomatika i Telemekhanika, 19, No. 10 (1958).
4. N. V. Pozin, "The interference-stability of pulse-width and pulse-time telemetering for strong fluctuation interference," Avtomatika i Telemekhanika, 20, No. 2 (1959).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

POTENTIAL AND REAL NOISE STABILITY OF MULTI-CHANNEL RADIOTELEMETERING SYSTEMS WITH TIME DIVISION OF CHANNELS UNDER WEAK FLUCTUATING NOISE

A. F. Fomin

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1405-1415, October, 1961
Original article submitted February 8, 1961

The potential noise stability is determined for various methods of transmitting multi-channel telemetering information with time division of the channels. A comparison is made of the various transmission methods for real and potential noise stability.

Introduction

Although at present there is a sufficiently large amount of work both on potential [1, 3-5], and on real [2, 6, 7] noise stability, yet not all radiotelemetering systems (RTS) with time division of the channels (TDC) have been examined with respect to potential noise stability, and there has not been presented a comparison of potential and real noise stability for many systems.

In the present work, there will be used as a criterion for evaluating potential noise stability with weak fluctuating noise the value of the mean square error measured at the receiver output, taken for the entire range of parameter values, which is found from the formula [1, 4]

$$\delta_p^2 = \frac{\sigma^2}{8\int\limits_{-T/2}^{T/2}\left[\frac{\partial A(\lambda, t)}{\partial \lambda}\right]} = \frac{\sigma^2}{2I}. \tag{1}$$

As a criterion for evaluating the noise stability of real receivers there will be used the value of the mean square error of the measurement, also taken for the entire range of parameter values, which we will determine from the formula

$$\delta_r^2 = \frac{1}{8}\left(\frac{V_p}{V_s}\right)^2_{out}. \tag{2}$$

In the present work the ratios $(V_s/V_p)_{out}$ are utilized for various systems, as given by Nichols and Rauch [2] and other authors [6].

When determining the real noise stability of radiotelemetering systems with time division of the channels it is assumed that the commutator of the receiving apparatus is rigidly synchronized with the commutator of the transmitting apparatus, and that in the output of each channel a low frequency filter (LFF) with a frequency cutoff $\Delta F = F_{max}$ is applied.

We use the following designations: $A(\lambda, t)$ is the signal which is a function of time and the parameter being measured; F_1 is the interrogation cycling frequency of the parameters being measured; ΔF is the bandwidth of the receiver channel; F_s is the bandwidth of the video channel; F_{max} is the maximum frequency of the parameter being measured; f_D is the maximum deviation of the carrier frequency; M is the coefficient of amplitude modulation of

the carrier; N is the number of channels; T is the interrogation cycling period of the parameters being measured

$$\left(T = \frac{1}{F_i} = \frac{1}{2F_{max}}\right);$$

V_E is the effective voltage of the unmodulated carrier; V_m is the maximum amplitude of the carrier voltage; λ is the parameter being measured in the interval ± 1 (the value in this interval is equally probable); α_T is the utilization factor for the channel time ($\alpha_T = T/T_c$); δ is the mean-square error; σ is the unit intensity of the fluctuating noise ($v/\sqrt{cps}$); ω_0 is the angular frequency of the carrier; ω_D is the maximum angular deviation of the carrier; τ_0 is length of a pulse; $\Delta\tau$ is the maximum deviation of a pulse; τ_F is the length of the rising (falling) front of a pulse.

1. Pulse Amplitude Modulation-Amplitude Modulation System (PAM-AM)

Potential noise stability. For a system with PAM-AM modulation the signal can be written in the form

$$A(\lambda, t) = \begin{cases} V_m\,[1 + M\lambda]\sin(\omega_0 t + \varphi) & \text{for } -\frac{\tau_0}{2} < t < \frac{\tau_0}{2}, \\ 0 & \text{for } \frac{\tau_0}{2} < t < -\frac{\tau_0}{2}. \end{cases}$$

For this signal we obtain

$$A'(\lambda, t) = V_m M \sin(\omega_0 t + \varphi) \text{ for } -\frac{\tau_0}{2} < t < \frac{\tau_0}{2},$$

whence

$$I = \int_{-\tau_0/2}^{\tau_0/2} \left[\frac{\partial A(\lambda, t)}{\partial \lambda}\right]^2 dt = V_m^2 M^2 \int_{-\tau_0/2}^{\tau_0/2} \sin^2(\omega_0 t + \varphi)\, dt = \frac{1}{2} V_m^2 M^2 \tau_0.$$

When calculating the integral it is assumed [1] that for

$$T \gg \frac{2\pi}{\omega_0} \quad \sin^2(\omega_0 t + \varphi) = \frac{1}{2}.$$

We will use this assumption in the future without additional reservations.

Substituting I in formula (1) and taking M = 1 we obtain

$$\delta_p^2 = \frac{\sigma^2}{8I} = \frac{\sigma^2}{V_m^2 \tau_0}. \tag{3}$$

For a comparison with other systems it is convenient to convert from the power (maximum amplitude) of the signal in a pulse to the average power (average effective voltage) of the transmitter. When M = 1 for an N-channel system we will have approximately

$$V_m^2 = 2V_E^2 \frac{T}{\tau_0 N}.$$

Substituting this value in formula (3) we obtain

$$\delta_p^2 = \frac{\sigma^2}{8V_E^2 T} N. \tag{4}$$

Real noise stability. On the basis of an examination of the signal and noise transmission processes in receiving apparatus it is possible to obtain the following expression [2]:

$$\frac{V_s}{V_p} = \frac{V_E}{\sqrt{2}\sigma\sqrt{\Delta F}}\,\frac{1}{\sqrt{N}}\,. \tag{5}$$

The mean-square error of the measurements will, in addition, be found from the expression

$$\delta_r^2 = \frac{N\sigma^2\Delta F}{4\,V_E^2}\,. \tag{6}$$

The ratio of the measurement errors during reception in real and ideal receivers will be

$$P_e^2 = \frac{\delta_r^2}{\delta_p^2} = \frac{\Delta F}{F_{max}}\,. \tag{7}$$

Thus a real receiver with the PAM-AM system provides the potential noise stability only in the case where demodulation of a pulse series is accomplished with the help of an ideal LFF with a frequency cutoff equal to the maximum information frequency ($\Delta F = F_{max}$).

2. Pulse Amplitude Modulation-Frequency Modulation System (PAM-FM)

Potential noise stability. For a system of the PAM-FM type the signal can be represented in the form

$$A(\lambda, t) = \begin{cases} V_m \cos\left[(\omega_0 + \omega_D\lambda)\,t + \varphi_0\right] & \text{for } -\tau_0/2 < t < \tau_0/2, \\ V_m \cos(\omega_0 t + \varphi_1) & \text{for } -\tau_0/2 < t < -\tau_0/2. \end{cases} \tag{8}$$

The expression for the error in a multi-channel system ($\tau_0 = T/N\alpha_T$, $V_m = \sqrt{2}V_E$) will be

$$\delta_p^2 = \frac{3\sigma^2}{2V_E^2\,\omega_D^2\,T^3}\,N^3\alpha_T^3\,. \tag{9}$$

Real noise stability. The expression for the mean square error in a general form, obtained on the basis of Nichols and Rauch's data [2], is

$$\delta_r^2 = \frac{\alpha_T^2 N^2\sigma^2 F_s F_i \Delta F}{V_E^2\,\omega_D^2}\,. \tag{10}$$

The ratio of the errors determined from the receiving condition in a real and an ideal (per Kotel'nikov) receiver is found from the expression

$$P_e^2 = \frac{\delta_r^2}{\delta_p^2} = \frac{2}{3}\,\frac{F_s\,\Delta F T^2}{N\alpha_T} = \frac{1}{3}\,\frac{F_s\Delta F}{F_m}\,\tau_0\,. \tag{11}$$

In the special case when $F_s = 3/\tau_0$ and $\Delta F = F_{max}$ we get $P_e^2 = 1$.

It should be noted that a calculation of video bandwidth according to the formula $F_s = 3/T_0$ obviously gives a somewhat higher value for F_s. If we take $F_s < 3/\tau_0$, then we find that the real noise stability according to Nichols and Rauch is higher than the potential according to Kotel'nikov, which is very unlikely. However, a detailed discussion of this question is beyond the scope of the present article.

3. Pulse Amplitude Modulation-Phase Modulation System (PAM-PM)

Potential noise stability. The signal can be represented in the form

$$A(\lambda, t) = \begin{cases} V_m \cos(\omega_0 t + \Phi_D\lambda) & \text{for } -\tau_0/2 < t < \tau_0/2, \\ V_m \cos(\omega_0 t + \varphi_0) & \text{for } \tau_0/2 < t < -\tau_0/2, \end{cases} \tag{12}$$

where $\Phi_D = f_D/F_s$ is the maximum phase deviation of the carrier frequency.

On the basis of formula (1), taking into account that $\tau_0 = T / Nd_T$, $\Phi = f_D/F_s$, and $V_m = \sqrt{2}V_E$, we get

$$\delta_p^2 = \frac{N\alpha_T \sigma^2 F_s^2}{8V_E^2 f_D^2 T}. \tag{13}$$

Real noise stability. The mean square measurement error at the output of a real receiver is found from the expression [2]

$$\delta_r^2 = \frac{\alpha_T N\sigma^2 F_s^2 \Delta F}{4V_E^2 f_D^2}. \tag{14}$$

The ratio of the errors found from formulas (13) and (14) will be

$$P_e^2 = \frac{\delta_r^2}{\delta_p^2} = 2\Delta FT. \tag{15}$$

In the special case when $\Delta F = F_{max}$

$$P_e^2 = 1.$$

From the comparison given it is seen that the noise stability of a real receiver for the PAM-PM system with the proper choice of video bandwidth in the receiver and with an ideal LFF having a cutoff $\Delta F = F_{max}$ in the PAM demodulator likewise achieves the potential noise stability.

4. Pulse Position Modulation-Amplitude Modulation System (PPM-AM)

Potential noise stability. The signal for such a modulation can be represented by the expression

$$A(\lambda, t) = V_m F\left[t - \left(t_c + \frac{\Delta\tau}{2}\lambda\right)\right]\cos(\omega_0 t + \varphi). \tag{16}$$

We will find the mean square error by superposing on the signal a low intensity noise during reception by an ideal receiver. To determine the derivative $A'(\lambda, t)$ we will substitute the variables $t^* = t - \left(t_c + \frac{\Delta\tau}{2}\lambda\right)$:

$$A'(\lambda, t) = \frac{\partial F}{\partial t^*}\frac{dt^*}{d\lambda} = V_m \frac{\partial F}{\partial t^*}\frac{\Delta\tau}{2}\cos(\omega_0 t + \varphi),$$

$$I = \overline{TA'^2(\lambda, t)} = \frac{1}{8}V_m^2 \Delta\tau^2 \int_{-T/2}^{T/2}\left[\frac{\partial F}{\partial t^*}\right]^2 dt.$$

The mean square measurement errors are

$$\delta_p^2 = \frac{\sigma^2}{8\overline{TA'^2(x, t)}} = \frac{\sigma^2}{\Delta\tau^2 V_m^2\, \overline{T\left[\frac{\partial F(t^*)}{\partial t^*}\right]^2}}. \tag{17}$$

It is evident from these formulas that the error will be less, the greater $\Delta\tau$ and the greater the unit energy of the fluctuation $\partial F(t^*)/\partial t^*$, i.e., the error depends on the pulse shape, or more exactly, on the type of rise in the leading and trailing edges of the pulse.

If it is assumed that the rise and fall of the pulse occur instantly, then $\partial F^* / \partial t^* = \infty$ and $\delta_p^2 = 0$.

We will consider the case of a triangular pulse of $2\tau_F$ duration and with a linear rise and fall of the wavefronts. Then

$$\frac{\partial F^*}{\partial t^*} = \frac{1}{\tau_F}, \quad \overline{T\left(\frac{\partial F^*}{\partial t^*}\right)^2} = \int_{-\tau_F}^{\tau_F}\frac{1}{\tau_F^2}dt = \frac{2}{\tau_F}.$$

Substituting the expression obtained into formula (17) and converting from the signal power in the pulse to the average power of the transmitter, for an N-channel system we find

$$V_{m\Delta}^2 = 3V_E^2 \frac{T}{\tau_F N}, \quad \Delta\tau = \frac{T}{N\alpha_T}.$$

As a result we get

$$\delta_{p\Delta}^2 = \frac{\sigma^2 N^3 \alpha_T^2 \tau_F^2}{6 V_E^2 T^3}. \tag{18}$$

If it is assumed that the envelope of the signal pulse is represented by a function of the form

$$F(t) = \frac{\sin \Omega_s t}{\Omega_s t}, \tag{19}$$

then the expression for the mean square error can be obtained in the form [1]

$$\delta_{p\Omega}^2 = \frac{3\sigma^2 N^3 \alpha_T^2}{2 V_E^2 T^3 \Omega_s^2}. \tag{20}$$

A comparison of formulas (19) and (20) shows that when the parameters σ, N, α_T, V_E, T, and $F_s = 1/\tau_F$ are identical the system using pulses with the envelope $F(t) = \sin \Omega_s t / \Omega_s t$ has a somewhat higher noise stability.

Real noise stability. We will determine the mean square measurement errors at the output of a real receiver operating on the leading edge of the pulse, which increases according to a linear law. The receiver operates when the pulse amplitude attains a certain value.

The mean square displacement value of a triangular pulse due to noise is found from the formula [5]

$$\overline{\Delta\tau_p^2} = \frac{\sigma^2}{V_m^2 F_s}. \tag{21}$$

This pulse shift due to noise will give the following error in the determination of the parameter λ:

$$\delta_{r\Delta}^2 = \frac{\overline{\Delta\tau_p^2}}{(\Delta\tau/2)^2} = \frac{\sigma^2}{V_m^2 F_s \Delta\tau^2}$$

(when λ is changed by unity the pulse is shifted by $\Delta\tau/2$).

Converting to the average transmitter power for a multi-channel system, we obtain

$$\delta_{r\Delta}^2 = \frac{\sigma^2 N^3 \alpha_T^2}{3 V_E^2 T^3 F_s^2}. \tag{22}$$

It should be noted that such mean square measurement errors will appear while receiving one-sided Pulse Width Modulation (PWM) signals on a real receiver.

The ratio of the measurement errors during the real and ideal reception of PPM-AM signals on the assumption of a linear rise in the wavefront of the pulse will be equal to

$$P_{e\Delta}^2 = \frac{2}{F_s \tau_F}. \tag{23}$$

If $F_s = 1/\tau_F$, then $P_e^2 = 2$.

The mean square measurement error at the output of a real receiver while employing limiting short pulses (19) according to the data of Nichols and Rauch is found from the expression

$$\delta^2_{I\Omega} = \frac{4}{25}\,\frac{\sigma^2 N^3 \alpha^2_T F_i^2 \Delta F}{F_s^2\, V_E^2} \tag{24}$$

From the ratio of Eq. (24) to (20) we get

$$p^2_{e\Omega} = 2\,\frac{\Delta F}{F_{max}} = 2.$$

From the comparison given it is evident that the real noise stability of this reception method for various pulse shapes will be $\sqrt{2}$ times lower than the potential noise stability of the transmission method.

These facts agree completely with the facts presented by V. A. Kotel'nikov in [1]. In this work it is shown that the potential noise stability of PPM-AM signals can be completely realized only when the detection of the time position of a pulse is at the maximum pulse amplitude.

5. Pulse Position Modulation-Frequency Modulation (PPM-FM)

In the work of Nichols and Rauch [2] it is indicated that the use of FM and PM with PPM gives no advantages over PPM-AM because the basic merit of PPM is that the transmitting apparatus is only switched on for a small time interval. We will examine this situation with respect to the potential noise stability.

For systems of the PPM-FM type which utilize pulses with envelopes described by expression (19) the signal can be represented in the form

$$A(\lambda,t) = \begin{cases} V_m \cos\left\{\left[\omega_0 + 2\omega_D \dfrac{\sin \Omega_s\left[t-\left(t_c+\frac{\Delta\tau}{2}\lambda\right)\right]}{\Omega_s\left[t-\left(t_c+\frac{\Delta\tau}{2}\lambda\right)\right]}\right] t + \varphi\right\} \\ \quad \text{for } t_c - \dfrac{\pi}{\Omega_s} < t < t_c + \dfrac{\pi}{\Omega_s}, \\ V_m \cos(\omega_0 t + \varphi_2) \quad \text{for } t_c + \dfrac{\pi}{\Omega_s} < t < t_c - \dfrac{\pi}{\Omega_s}. \end{cases} \tag{25}$$

For this signal we find

$$A'(\lambda,t) =$$

$$= -V_m \omega_D \Delta\tau t \frac{\partial}{\partial t}\left(\frac{\sin \Omega_s t}{\Omega_s t}\right) \sin\left\{\left[\omega_0 + 2\omega_D \frac{\sin \Omega_s\left[t-\left(t_c+\frac{\Delta\tau}{2}\lambda\right)\right]}{\Omega_s\left[t-\left(t_c+\frac{\Delta\tau}{2}\lambda\right)\right]}\right] t + \varphi\right\}.$$

For the calculation of the error we determine the integral

$$I = \int_{-T/2}^{T/2} \left[\frac{\partial A(\lambda,t)}{\partial \lambda}\right]^2 dt = \frac{V_m^2 \omega_D^2 \Delta\tau^2}{2} \int_{-T/2}^{T/2} \frac{\partial}{\partial t}\left[\frac{\sin \Omega_s t}{\Omega_s t}\right]^2 t^2\, dt =$$

$$= \frac{V_m^2 \omega_D^2\, \Delta\tau^2}{2}\,\frac{2\pi}{\Omega_s}.$$

Substituting the value of I in (1) and taking into account that $V_m = \sqrt{2} V_E$ and $\Delta\tau = T/N\alpha_T$, we obtain

$$\delta^2_{p\Omega} = \frac{\sigma^2 \Omega_s N^2 \alpha_T^2}{16\pi V_F^2 \omega_D^2 T^2}. \tag{26}$$

We will make a comparison of the PPM-AM and PPM-FM systems for the condition of the same average transmitter power V_E^2, the same speed of operation T, number of channels N and α_T, the same PPM signal parameters, and unit noise intensity σ in the communication channel.

To this end we will take the ratio of Eqs. (20) and (26):

$$\frac{\delta_p^2\,(\text{PPM-AM})}{\delta_p^2(\text{PPM-FM})} = 24\tau\left(\frac{\omega_d}{\Omega_s}\right)^2 \frac{N}{T\Omega_s}. \tag{27}$$

From (27) it is seen that the PPM-FM system in comparison with the PPM-AM system provides a much higher potential noise stability only in case the frequency modulation index is larger, i.e.,

$$\left(\frac{\omega_D}{\Omega_s}\right)^2 > \frac{T\,F_s}{12\,N}. \tag{28}$$

It is easily shown that the fulfillment of this inequality requires a high index of frequency modulation which is practically unrealizable when using short pulses.

Thus the PPM-FM system is actually worse with respect to noise stability than the PPM-AM system for the continuous transmission of signals.

6. Pulse Width Modulation-Amplitude Modulation (PWM-AM)

In radiotelemetry systems one-sided PWM is the most widely used. Therefore we will compare systems with respect to the real and potential noise stability on the assumption that a one-sided PWM is used in which the leading edge of the pulse is rigidly clamped during reception.

Potential noise stability. For such a system the signal can be represented in the form

$$A(\lambda, t) = \begin{cases} V_m F\left\{t - \left[t_c + \frac{\Delta\tau}{2}(1+\lambda)\right]\right\}\cos(\omega_0 t + \varphi) \\ \text{for } t_c < t < t_c + \frac{\Delta\tau}{2}(1+\lambda), \\ 0 \\ \text{for } t_c > t > t_c + \frac{\Delta\tau}{2}(1+\lambda). \end{cases}$$

For simplicity we will assume that the trailing edge of the pulse falls off according to a linear law and the duration of the edge is τ_F. In this case

$$I = \frac{V_m^2}{8}\Delta\tau^2 \int_{-\tau_F/2}^{\tau_F/2} \frac{1}{\tau_F^2}\,dt = \frac{V_m^2\,\Delta\tau^2}{8\tau_F}.$$

The mean square measurement error will be determined by the expression

$$\delta_p^2 = \frac{\sigma^2\tau_F}{V_m^2\Delta\tau^2}. \tag{29}$$

It can be shown that for two-sided PWM the error will be $\sqrt{2}$ times greater.

Converting to the average effective signal voltage of the transmitter

$$V_m^2 = 4V_E^2\frac{T}{\Delta\tau N}$$

(where $\Delta\tau/2$ is the average pulse length) and substituting $\Delta\tau = T/N\alpha_T$, we obtain

$$\delta_p^2 = \frac{\sigma^2\tau_F N^2\alpha_T}{4V_E^2T^2}. \tag{30}$$

Real noise stability. The signal-to-noise ratio at the output of the ith channel of a multi-channel receiver is found from the expression [2]

$$\frac{V_s}{V_p} = \frac{1}{N}\sqrt{\frac{F_s}{\alpha_T F_i}}\,\frac{V_E}{\sqrt{2}\,\sigma\sqrt{\Delta F}}\,.$$

Moreover, in a real receiver there will be the error

$$\delta_r^2 = \frac{N^2\alpha_T\sigma^2 F_i\,\Delta F}{4V_E^2 F_s}\,. \tag{31}$$

The ratio of the errors for real and ideal reception will be equal to

$$P_e^2 = \frac{\Delta F}{F_s\tau_F}\,T. \tag{32}$$

If $F_s = 0.5/\tau_F$, $\Delta F = F_{max}$, then $P_e^2 = 1$.

Thus it can be considered that a real receiver of PWM-AM signals achieves in practice the potential noise stability of this method when the leading edge of the pulse is rigidly fixed and

$$F_s = \frac{0.5}{\tau_F},\ \Delta F = F_{max}.$$

7. Pulse Width Modulation-Frequency Modulation (PWM-FM)

Potential noise stability. If it is assumed that the frequency change takes place according to a linear law, then the signal of such a system can be represented in the form

$$A(\lambda, t) = \begin{cases} V_m \sin(\omega_1 t + \varphi_1) & \text{for } 0 < t < t_c, \\ V_m \sin\left\{\left[\omega_1 + \frac{2\omega_D}{\tau_F}(t - t_c)\right]t + \varphi_2\right\} & \text{for } t_c < t < t_c + \tau_F, \\ V_m \sin(\omega_2 t + \varphi_3) & \text{for } t_c + \tau_F < t < t_2, \\ V_m \sin\left\{\left[\omega_2 - \frac{2\omega_D}{\tau_F}(t - t_2)\right]t + \varphi_4\right\} & \text{for } t_2 < t < t_2 + \tau_F, \end{cases} \tag{33}$$

where $t_2 = t_c + \tau_F + \frac{\Delta\tau}{2}(1 + \lambda)$; t_c is the beginning of the leading edge of the pulse. For this signal (considering that $V_m = \sqrt{2}V_E$) we get

$$A'(\lambda, t) = \sqrt{2}V_E\,\frac{\Delta\tau\omega_D}{\tau_F}t\cos\left\{\left[\omega_2 - \frac{2\omega_D}{\tau_F}(t - t_2)\right]t + \varphi_4\right\}$$

for $t_2 < t < t_2 + \tau_F$.

Hence

$$I = \int_{-\tau_F/2}^{\tau_F/2}\left[\frac{\partial A(\lambda, t)}{\partial\lambda}\right]dt = \frac{V_E^2\,\Delta\tau^2\omega_D^2}{12}\,\tau_F.$$

The mean square measuring error for multi-channel transmission ($\Delta\tau = T/N\alpha_T$) will be determined by the expression

$$\delta_P^2 = \frac{3\pi^2 N^2 \alpha_T^2}{2V_E^2 T^2 \omega_D^2 \tau_F}. \tag{34}$$

Real noise stability. The signal-to-noise ratio at the output of a real PWM-FM receiver is determined by the expression [2]

$$\frac{V_s}{V_m} = \frac{\sqrt{6} f_D}{\alpha_T N \sqrt{F_s}\; F_i} \frac{V_E}{\sqrt{2}\sigma\sqrt{\Delta F}},$$

from which we will find the mean square error

$$\delta_R^2 = \frac{\alpha_T^2 N^2 \sigma^2 F_s\; F_i\, \Delta F}{24\, f_D^2 V_E^2}. \tag{35}$$

The ratio of the errors for reception on real and ideal (per Kotel'nikov) receivers will be equal to

$$P_e^2 = 1.8\, F_s \tau_F \frac{\Delta F}{F_{max}}. \tag{36}$$

If

$$F_s = \frac{1.8}{\tau_F};\; \Delta F = F_{max}, \text{then } P_e^2 = 1.1.$$

Obviously, as with PAM-FM systems, a calculation of the bandwidth according to the formula $F_s = 1.8/\tau_F$ gives a somewhat oversized value for F_s.

8. Pulse Width Modulation-Phase Modulation (PWM-PM)

Potential noise stability. Proceeding from the same premises as for the development of the type PWM-FM telemetering system, the signal for a PWM-PM system can be represented in the form

$$A(\lambda, t) = \begin{cases} V_m \sin(\omega_0 t + \varphi_1) & \text{for } 0 < t < t_c, \\ V_m \sin\left[\omega_0 t + \varphi_1 + \frac{2\Phi_D}{\tau_F}(t - t_c)\right] & \text{for } t_c < t < t_c + \tau_F, \\ V_m \sin(\omega_0 t + \varphi_2) & \text{for } t_c + \tau_F < t < t_2, \\ V_m \sin\left[\omega_0 t + \varphi_2 - \frac{2\Phi_D}{\tau_F}(t - t_2)\right] & \text{for } t_2 < t < t_2 + \tau_F, \end{cases} \tag{37}$$

where $t_2 = t_c + \tau_F + \frac{\Delta\tau}{2}(1 + \lambda)$, t_c is the beginning of the leading pulse edge.

Making a change of variables and substituting $V_m = \sqrt{2}V_E$, we get

$$A'(\lambda, t) = \sqrt{2}\, V_E \frac{\Phi_D}{\tau_F} \Delta\tau \cos\left[\omega_0 t + \varphi_2 - \frac{2\Phi_D}{\tau_F}(t - t_c)\right] \text{ for } t_2 < t < t_2 + \tau_F.$$

For the desired integral we have

$$I = \frac{V_E^2 \Phi_D^2 \Delta\tau^2}{\tau_F}.$$

Moreover, the integration is taken over the limits from $-\tau_F/2$ to $+\tau_F/2$ because in the remaining time $A'(\lambda, t) = 0$.

The mean square error when receiving on an ideal receiver will be found from

$$\delta_{\mathrm{P}}^2 = \frac{\sigma^2 \tau_{\mathrm{F}} N^2 \alpha_{\mathrm{T}}^2}{8 V_{\mathrm{E}}^2 \Phi_{\mathrm{D}}^2 T^2}. \tag{38}$$

Real noise stability. To find the error in a real PWM-PM receiver we will use the signal-to-noise ratio at the receiver output presented in [2]:

$$\frac{V_s}{V_{\mathrm{P}}} = \frac{\sqrt{2}\,\Phi_{\mathrm{D}}}{N\alpha_{\mathrm{T}}} \left(\frac{F_s}{F_t}\right)^{1/2} \frac{V_{\mathrm{E}}}{\sqrt{2}\,\sigma\sqrt{\Delta F}}.$$

Hence

$$\delta_{\mathrm{r}}^2 = \frac{\sigma^2 N^2 \alpha^2 F_t \Delta F}{8 V_{\mathrm{E}}^2 \Phi_{\mathrm{D}}^2 F_s}. \tag{39}$$

The ratio of errors when receiving on an ideal and a real receiver (for identical signal parameters) will be

$$P_{\mathrm{e}}^2 = \frac{\Delta F T}{F_s \tau_{\mathrm{F}}} = \frac{\Delta F}{F_s \tau_{\mathrm{F}} 2F_{\max}}. \tag{40}$$

If $F_s = 0.5/\tau_{\mathrm{F}}$, $\Delta F = F_{\max}$, then $P_{\mathrm{e}}^2 = 1$.

Consequently, for the conditions enumerated above in a real receiver of PWM-PM signals it is also possible to realize the potential noise stability.

SUMMARY

The results of a comparison among various types of multi-channel radiotelemetering systems with time division of the channels which was made with respect to real and potential noise stability show that for certain conditions the noise stability of almost all systems approaches the potential.

One of the necessary conditions for the real noise stability of systems with PAM and PWM to approach the potential is the use in demodulating of a pulse series of a low-frequency filter having a frequency cutoff equal to the maximum frequency of the parameter being measured ($\Delta F = F_{\max}$). The second necessary condition for achieving the potential noise stability in the group of systems is the optimum selection of the bandwidth passed by the video channel.

The real noise stability of a PPM-AM type system, even when the conditions given above are observed, is worse than the potential by approximately $\sqrt{2}$ times. Since real filters cannot have the infinitely great sharpness of cutoff, i.e., the width of the noise band (ΔF) is always greater than $F_{\max}$, then the real noise stability of the systems will always be somewhat worse than the potential.

It should also be noted that when in a radiotelemetering system with time division of the channels a secondary frequency modulation (PAM-FM, PWM-FM) is used, formula (1) gives a somewhat oversized value of δ_{P}. This can be explained by the fact that according to [1] for all pulse modulation systems the noise intensity at the output of an ideal receiver has a uniform spectrum. However, a detailed discussion of this question is beyond the scope of the article.

The author expresses deep appreciation to A. P. Manovtsev for valuable advice and interest in the work.

LITERATURE CITED

1. V. A. Kotel'nikov, Theory of Potential Noise Stability (Gosenergoizdat, 1956).
2. M. C. Nichols and L. L. Rauch, Radiotelemetry [Russian translation] (Izd. inostr. liter., 1958).
3. V. A. Kashirin, "Noise stability of frequency modulation systems," Avtomatika i telemekhanika, 18, No. 6 (1957).
4. V. A. Kashirin and G. A. Shastova, "Noise stability in transmission of telemetering signals over a channel with fluctuating noise," Avtomatika i telemekhanika, 19, No. 8 (1958).
5. V. A. Kashirin, "Optimal parameters for noise stability in telemetering systems," Avtomatika i telemekhanika, 20, No. 2 (1959).
6. S. V. Borodich, Multi-Channel Radio Relay Communication Lines [in Russian] (Svyaz'izdat, 1953).
7. V. D. Smirnov, Principles of Ultrashort-Wave Radio Communication [in Russian] (Izd. lit. po voprosam svyazi i radio, 1957).

DETERMINATION OF DISTURBANCE CHARACTERISTICS IN AIRCRAFT – AUTOPILOT SYSTEMS

A. S. Uskov

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10, pp. 1416-1418, October, 1961
Original article submitted March 27, 1961

A method for determining the mathematical expectation, the autocorrelation function, and the disturbance spectrum in aircraft–autopilot systems is presented. The described method is also valid for arbitrary linear controlled systems and for automatic control systems.

The problem of the statistical analysis of disturbance characteristics in a linear aircraft–autopilot system can be stated in the following manner.

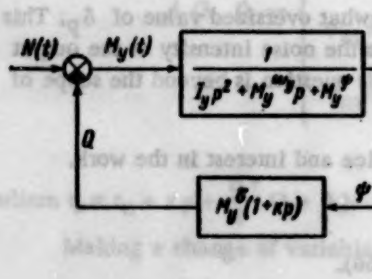
The dynamic characteristics of the aircraft and the autopilot and certain given realizations of steady-state random processes at their inputs and outputs are assigned. It is necessary to find the equivalent disturbance characteristics–the mathematical expectation, the autocorrelation function, and the spectrum.

By the term equivalent disturbance, we shall designate a disturbance representing the over-all action of disturbances that affect the controlled quantity and are reduced to a certain given point of the system, which is generally arbitrarily selected.

Let us consider the flight of an aircraft with an autopilot on a certain course. In the first approximation, the linearized equation describing the dynamics of the aircraft – autopilot system can be written in the following form[1]:

$$(J_y p^2 + M_y^{\omega_y} p + M_y^{\psi})\,\psi = -Q + N, \qquad Q = M_y^{\delta}(1 + kp)\,\psi, \tag{1}$$

where J_y is the moment of inertia of the aircraft with respect to its vertical axis Y (a right-handed coordinate system which is fixed to the aircraft with the coordinate origin at its center of gravity is used); $M_y = -Q + N$ is the total yawing moment, which acts on the aircraft with respect to the Y axis; $M_y^{\psi} = \dfrac{\partial M_y}{\partial \psi}$ is the derivative which characterizes the yawing stability; $M_y^{\omega_y} = \dfrac{\partial M_y}{\partial \omega_y}$ is the damping-in-yaw derivative; $M_y^{\delta} = \dfrac{\partial M_y}{\partial \delta}$ is the derivative of the moment with respect to the rudder deflection; ψ is the sideslip angle; $\omega_y = p\psi$; $\delta = (1 + kp)\,\psi$ is the rudder deflection angle, $\underline{k}$ is the transfer constant in front of the derivative; N(t) is the yawing disturbance, which can be caused by wind gusts, air vortices, the effect of banking motion on the motion along the flight path, random changes in the air temperature and pressure along the flight path, etc.; and p = d/dt is the differentiation operator.



The block diagram of the system is shown in the figure.

In this case, we shall consider the disturbance N(t) to be the equivalent disturbance which acts at the system's input. We shall assume that the transfer function of the aircraft

$$\Phi = \frac{1}{J_y p^2 + M_y^{\omega_y} p + M_y^{\psi}}$$

and the transfer function of the autopilot

$$\Phi_a = M_y^\delta (1 + kp).$$

are known.

Moreover, it is assumed that the curve of the yawing moment M_y acting on the aircraft is known in a sufficiently large time interval $T + \tau$ [2-4]. We shall find the mathematical expectation EN(t), the autocorrelation function $R_N(\tau)$, and the spectrum $S_N(\omega)$ of the disturbance N(t) that acts at the system's input.

For the system under consideration

$$N(t) = \int_0^\infty M_y (t - \vartheta)\, k(\vartheta)\, d\vartheta, \tag{2}$$

where k(t) is the pulse transfer function which corresponds to the transfer function $1 + \Phi\Phi_a$.

By using Eq. (2), we obtain [4]

$$R_N(\tau) = \int_0^\infty R_{M_y N} (\tau - \vartheta)\, k(\vartheta)\, d\vartheta, \tag{3}$$

$$R_{M_y N}(\tau) = \int_0^\infty R_{M_y} (\tau + \vartheta)\, k(\vartheta)\, d\vartheta, \tag{4}$$

where

$$R_N(\tau) = \frac{1}{T} \int_0^T N'(t)\, N'(t - \tau)\, dt,$$

$$R_{M_y N}(\tau) = \frac{1}{T} \int_0^T M'_y(t)\, N'(t - \tau)\, dt,$$

$$R_{M_y}(\tau) = \frac{1}{T} \int_0^T M'_y(t)\, M'_y(t - \tau)\, dt,$$

and N'(t) and $M'_y(t)$ are centered inputs, which are determined by the equations

$$N'(t) = N(t) - EN(t), \quad M'_y(t) = M_y(t) - EM_y(t),$$

where E is the symbol denoting the mathematical expectation.

The mathematical expectation and the disturbance spectrum are determined by using the equations

$$EN(t) = EM_y(t) \int_0^\infty k(\vartheta)\, d\vartheta, \tag{5}$$

$$S_N(\omega) = 2 \int_0^\infty R_N(\tau) \cos \omega\tau\, d\tau. \tag{6}$$

We shall assume that $M_y(t)$ is an ergodic steady-state random process, by the statistical processing of which we obtain the following expressions:

$$EM_y(t) = 0, \tag{7}$$

$$R_{M_y}(\tau) = Ce^{-\alpha|\tau|}. \tag{8}$$

We shall write the pulse transfer function in the following form:

$$k(t) = \delta(t) + e^{\varepsilon t}(l \cos \omega_1 t + m \sin \omega_1 t), \tag{9}$$

where $\delta(t)$ is a delta-function, and

$$l = \frac{M_y^{\delta} k}{J_y}, \quad m = \frac{M_y^{\delta}}{J_y} \frac{2J_y - M_y^{\omega_y} k}{\sqrt{4J_y M_y^{\psi} - (M_y^{\omega_y})^2}},$$

$$\varepsilon = -\frac{M_y^{\omega_y}}{2J_y}, \quad \omega_1 = \sqrt{\frac{M_y^{\psi}}{J_y} - \frac{1}{4}\left(\frac{M_y^{\omega_y}}{J_y}\right)^2}.$$

By taking into account expressions (7) and (8) and by using Eqs. (4), (3), (5), and (6), we successively obtain the following statistical characteristics of the disturbance N(t):

$$EN(t) = 0, \tag{10}$$

$$R_N(\tau) = AR_{M_y}(\tau) + Ce^{\varepsilon|\tau|}(F \cos \omega_1 \tau - E \sin \omega_1 |\tau|), \tag{11}$$

$$S_N(\omega) = 2C\left\{\frac{A\alpha}{\alpha^2 + \omega^2} - \frac{F\varepsilon(\varepsilon^2 + \omega^2 + \omega_1^2) + E\omega_1(\varepsilon^2 - \omega^2 + \omega_1^2)}{[\varepsilon^2 + (\omega_1 + \omega)^2][\varepsilon^2 + (\omega_1 - \omega)^2]}\right\}, \tag{12}$$

where

$$A = \left[1 + \frac{m\omega_1 - l(\alpha + \varepsilon)}{(\alpha + \varepsilon)^2 + \omega_1^2}\right]\left[1 + \frac{m\omega_1 + l(\alpha - \varepsilon)}{(\alpha - \varepsilon)^2 + \omega_1^2}\right],$$

$$E = ab - df, \quad F = db + af,$$

$$a = \frac{\omega_1(l\varepsilon - m\omega_1)}{4\varepsilon(\varepsilon^2 + \omega_1^2)}, \quad b = \frac{l(\alpha + \varepsilon) - m\omega_1}{(\alpha + \varepsilon)^2 + \omega_1^2} + \frac{l(\alpha - \varepsilon) + m\omega_1}{(\alpha - \varepsilon)^2 + \omega_1^2},$$

$$d = \frac{4\varepsilon^3 - 2l\varepsilon^2 + m\varepsilon\omega_1 + \omega_1^2(4\varepsilon - l)}{4\varepsilon(\varepsilon^2 + \omega_1^2)},$$

$$f = \frac{l\omega_1 + m(\alpha + \varepsilon)}{(\alpha + \varepsilon)^2 + \omega_1^2} + \frac{-l\omega_1 + m(\alpha - \varepsilon)}{(\alpha - \varepsilon)^2 + \omega_1^2}.$$

Let us consider an example. Assume that J_y = 450 kg-m·sec^2, $M_y^{\omega_y}$ = 8500 kg-m·sec, M_y^{ψ} = 120000 kg-m, M_y^{δ} = 24000 kg-m, k = 1/3 sec, $EM_y(t) = 0$, $R_{M_y}(\tau) = Ce^{-0.5|\tau|}$.

By using Eqs. (5), (11) and (12), we successively obtain

$$EN(t) = 0,$$

$$R_N(\tau) = C[1.44\, e^{-0.5|\tau|} + e^{-9.44|\tau|}(0.0872 \cos 13.3\tau + 0.0731 \sin 13.3|\tau|)],$$

$$S_N(\omega) = C\left[\frac{1.44}{\omega^2 + 0.25} - \frac{0.29\,\omega^2 - 950}{\omega^4 - 0.18\,\omega^2 + 71\,000}\right].$$

The case considered above corresponds to an actual dynamic test of an aircraft model with autopilot in an aerodynamic wind tunnel. The total moment $M_y(t)$ acting on the aircraft was in this case realized by means of aerodynamic scales.

The characteristics of the yawing disturbance N(t) acting on the aircraft can also be found by using the results of flight tests if the means for realizing either the sideslip angle ψ or the rudder deflection angle δ are available to the experimenter. The characteristics of rolling and pitching disturbances can be determined in a similar manner.

LITERATURE CITED

1. V. S. Vedrov, Dynamic Aircraft Stability [in Russian] (Oborongiz, 1939).
2. B. N. Kutin, "Calculation of the autocorrelation function of steady-state random processes with respect to experimental data," Avtomatika i Telemekhanika, 18, No. 3 (1957).
3. A. E. Kharybin, "Analysis of errors in determining the average value of a random quantity and of its square that are due to the inaccuracy of the observation time," Avtomatika i Telemekhanika, 18, No. 4 (1957).
4. V. V. Solodovnikov, Introduction to the Statistical Dynamics of Automatic Control Systems [in Russian] (Gostekhizdat, 1952).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

INCREASING THE ACCURACY AND INTEGRATION TIME OF INTEGRATING AMPLIFIERS

I. B. Negnevitskii and S. B. Negnevitskii

(Moscow)

Translated from Avtomatika i Telemekhanika, Vol. 22, No. 10,
pp. 1419-1421, October, 1961
Original article submitted February 20, 1961

Wide use is being made today of integrating amplifiers with parallel feedback and with magnetic feedback in the case of magnetic amplifiers [1]. To assure accuracy and the possibility of integrating relatively low frequencies, the time constant of the system must be sufficiently large. This can be achieved either by increasing the capacitance in the feedback loop or by increasing the amplification factor of the amplifier. Each of these methods has its drawbacks.

Increasing the capacitance leads to an increase in the volume of the capacitors and a decrease of their leakage resistance. The latter leads to an increase in the fixed negative feedback and a corresponding decrease in the amplification factor, which sometimes nullifies the effect of the increase in capacitance.

An increase in the amplification factor usually requires an additional amplification stage, which leads to an increase in the drift voltage at the amplifier output, as well as to poorer stability of the system.

Cases are encountered in practice in which both possibilities have been exhausted and it is impossible to increase any further either the capacitance or the amplification factor, but the time constant of the system must be made several times as large.

The problem may also be formulated in another way. It is required to decrease the drift voltage at the output, at the same time keeping the same value for the time constant of the system. If in order to decrease the drift we decrease the amplification factor, then at the same time we decrease the time constant, and the integration errors will increase.

We present below a circuit for an integrating amplifier containing an additional auxiliary amplifier and making it possible to some degree to satisfy the above-mentioned contradictory requirements [2]. Both amplifiers may be either push-pull or single-ended amplifiers.

The figure below shows circuits for connecting an auxiliary amplifier for the case of parallel feedback, and for the case of magnetic feedback when the basic amplifier is magnetic. The amplifiers proper we shall first consider to be of electronic type.

The transfer function for parallel feedback (figure, a) is equal to

$$W(p)=\frac{k_{ub}}{1+R/R_b}\frac{1}{1+pT} \tag{1}$$

where T is the time constant of the system, equal to

$$T=\frac{\tau k_{ub}}{1+R/R_b}\left(1+\frac{1}{k_{ub}}+k_{ua}\right), \tag{2}$$

k_{ub} and k_{ua} are the voltage amplification factors of the basic and auxiliary amplifiers respectively; $\tau = RC$. For $1/k_{ub} \ll (1 + k_{ua})$ we have

$$T=\frac{\tau k_{ub}}{1+R/R_b}(1+k_{ua}). \tag{3}$$

For magnetic feedback the transfer function has the form

$$W(p)=\frac{U_{\text{out}}}{I_{\text{in}}}=\frac{k_F w_{\text{in}}(1+p\tau)}{1+pT}, \tag{4}$$

where $k_F = U_{out}/F_{in}$ is magnetomotive force amplification factor of the basic amplifier; $\tau = R_b C$; T is the time constant of the system, equal to

$$T=[R_{\text{b}}+(1+k_{\text{ua}})\,k_F w_{\text{b}}]\,C. \tag{5}$$

For $|p\tau| \ll 1$ and $R_b \ll (1+k_{ua})k_F w_b$, which is usually the case, we obtain

$$W(p)=\frac{k_F w_{\text{in}}}{1+pT},\quad T=Cw_{\text{b}}\,k_F\,(1+k_{\text{ua}}). \tag{6}$$

From expressions (3) and (6) it can be seen that the time constant of the system here is $(1 + k_{ua})$ times as big as in the known analogous circuits for integrating amplifiers without auxiliary amplifiers [1].

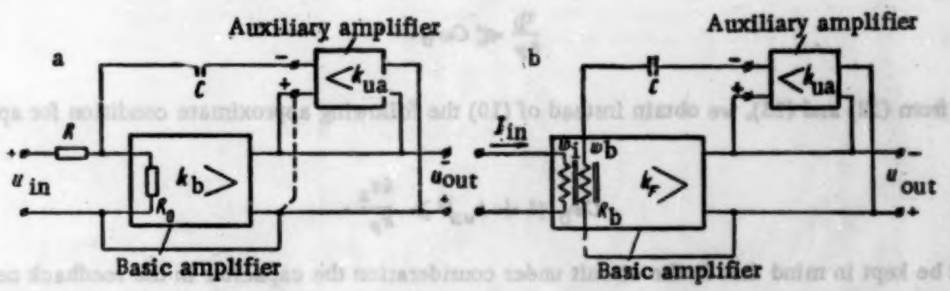
Now we shall consider the questions of the influence of drift, which are very important for integrating amplifiers. We shall find what the drift voltage U_{db} at the output of the basic amplifier will be when a drift voltage U_{da} appears at the output of the auxiliary amplifier. For this case, for example, with magnetic feedback, the transfer function is equal to

$$W_{\text{d}}(p)=\frac{U_{\text{db}}}{U_{\text{da}}}=\frac{k_F w_{\text{b}}\,pC}{1+pT}. \tag{7}$$

From (6) and (7) follows the expression, obvious in this case, for the basic amplifier drift seen at the input:

$$U_{\text{d in}}=U_{\text{da}}\frac{w_{\text{b}}}{w_{\text{in}}}Cp. \tag{8}$$

From (7) and (8) and from the circuit of figure b it can be seen that in the steady state a drift voltage U_{da} = = constant of the auxiliary amplifier will not cause any additional drift in the basic amplifier. Only the drift voltage of the basic amplifier will appear in the steady state, and this will not increase because of the introduction of the auxiliary amplifier.



If the drift voltage U_{da} changes comparatively rapidly, and $|pT| \gg 1$, then the expression (7) will approximately be

$$W(p)\approx\frac{1}{1+k_{ua}}, \tag{9}$$

i.e., in this case the drift at the output caused by the drift voltage of the auxiliary amplifier will be smaller by a factor of $1 + k_{da}$.

In order to obtain a maximum time constant of the system, the amplification factor k_{da} should be taken as large as possible. However, the value of k_{da} is limited by the linear range of the characteristic of the auxiliary

amplifier. In practice the maximum obtainable voltage (at the end of the linear zone) at the output of the basic amplifier lies between 10 and 100 volts. If we consider the maximum voltage at the output of the auxiliary amplifier to be equal to 100 volts, then it is obvious that the value of k_{ua} may lie between 10 and 1.0.

The load current i_2 of the auxiliary amplifier is the charging (discharging) current of the capacitance. For the integration times (frequencies) and capacitance values ordinarily encountered, this current and the output power of the auxiliary amplifier are comparatively small. For example, let C = 100 μf, the integration time be equal to 40 seconds, and the maximum output voltage of the auxiliary amplifier be 100 volts. Then if the nominal signal is suddenly connected at the input of the system we find approximately

$$i_2 = C\,\frac{du_a}{dt} \approx 0.25 \text{ ma},$$

and the maximum output power is $P_2 \approx 25$ mw. Thus, the auxiliary amplifier can be comparatively low-power and can have a comparatively high output resistance.

Let us now consider the question of the stability of the above-described integrating system. We shall assume that the basic and auxiliary amplifiers are both first-order magnetic elements, with time constants τ_b and τ_a respectively. Then, for example, for the case of magnetic feedback, the transfer function of the system will be equal to

$$W(p) = \frac{U_{out}}{I_{in}} = \frac{(1 + p\tau_a)\,w_{in}}{\frac{1}{k_F} + p\left[\frac{\tau_b + \tau_a}{k_F} + Cw_b\,(1 + k_{ua})\right] + p^2\left(\frac{\tau_b\tau_a}{k_F} + Cw_b\tau_a\right)} \tag{10}$$

From (10) it follows that the given system is stable for the assumptions we have made. Furthermore, it will be aperiodic if the condition

$$\left[\frac{\tau_b + \tau_a}{k_F} + Cw_b\,(1 + k_{ua})\right]^2 \geqslant \frac{4\tau_a}{k_F}\left(\frac{\tau_b}{k_F} + Cw_b\right). \tag{11}$$

is satisfied.

In practice the following relations are usually observed:

$$\frac{\tau_b + \tau_a}{k_F} \ll Cw_b\,(1 + k_{ua}), \tag{12}$$

$$\frac{\tau_b}{k_F} \ll Cw_b. \tag{13}$$

Starting from (12) and (13), we obtain instead of (10) the following approximate condition for aperiodicity of the system:

$$Cw_b\,(1 + k_{ua})^2 \geqslant \frac{4\tau_a}{k_F}. \tag{14}$$

It should be kept in mind that in the circuit under consideration the capacitor in the feedback network is operating at a higher voltage than in a circuit without an auxiliary amplifier. As is known, the leakage resistance r_{lk} of some types of capacitors, particularly paper capacitors, decreases as the voltage increases. The presence of a finite leakage resistance decreases the time constant of the system in the case, for example, of magnetic feedback, by a factor of

$$\left[1 + \frac{k_F w_b\,(1 + k_{ua})}{r_{lk}}\right].$$

As can be seen from figure a, the auxiliary and basic amplifiers do not have a common ground at the input and output. If a tube type auxiliary amplifier is used, this may cause the familiar technical difficulties. If magnetic amplifiers are used, there will of course be no such difficulties. However, if we consider that the amplifica-

tion factor k_{ua} is generally no greater than 10, it is obvious that even the tube type amplifier will work sufficiently well. Furthermore, if the connection of the auxiliary amplifier output is changed (see the dashed line on figure a), we will have the aforementioned common ground, while retaining the general principle of the integrating circuit just considered. The difference in this case will be merely that we should substitute k_{ua} for $1 + k_{ua}$ in the corresponding formulas mentioned above.

Experimental investigation of the above-described integrating amplifier completely corroborated the assumptions and relations mentioned in the article, both qualitatively and quantitatively. We investigated circuits, both with tube amplifiers and with magnetic amplifiers, with parallel and magnetic feedbacks. We give as an example some average parameters of one of the circuits in which we used magnetic amplifiers: $k_F = 500$, $w_b = 5{,}000$, $k_{ua} = 5$, $C = 60\ \mu f$, $\tau_b = 0.25$ second, $\tau_a = 0.05$ second. The introduction of the auxiliary amplifier increased the time constant practically by a factor of 6 and made it approximately equal to 300 seconds. The inequalities (12)-(14) are satisfied to such a degree that the time constants of both amplifiers could be considerably greater.

Finally, we wish to emphasize once more that the integrating circuit with an auxiliary amplifier is only supplementary to the known integration circuits, and its use is advisable in the cases indicated at the beginning of this article.

LITERATURE CITED

1. S. B. Negnevitskii, "The use of magnetic amplifiers in integrating circuits," Avtomatika i Telemekhanika, 20, No. 10 (1959).
2. I. B. Negnevitskii and S. B. Negnevitskii, Author's Certificate, "A magnetic amplifier integrator," No. 136906 (1960).

All abbreviations of periodicals in the above bibliography are letter-by-letter transliterations of the abbreviations as given in the original Russian journal. *Some or all of this periodical literature may well be available in English translation.* A complete list of the cover-to-cover English translations appears at the back of this issue.

Soviet Journals Available in Cover-to-Cover Translation

ABBREVIATION	RUSSIAN TITLE		TITLE OF TRANSLATION	PUBLISHER	TRANSLATION BEGAN Vol.	Issue	Year
AÉ	Atomnaya énergiya		Soviet Journal of Atomic Energy	Consultants Bureau	1	1	1956
Akust. zh.	Akusticheskii zhurnal		Soviet Physics – Acoustics	American Institute of Physics	1	1	1955
	Antibiotiki		Antibiotics	Consultants Bureau	4	1	1959
Astr(on). zh(urn).	Astronomicheskii zhurnal		Soviet Astronomy–AJ	American Institute of Physics	34	1	1957
Avto(mat). svarka	Avtomaticheskaya svarka		Automatic Welding	British Welding Research Association (London)		1	1959
	Avtomatika i Telemekhanika		Automation and Remote Control	Instrument Society of America	27	1	1956
	Biofizika		Biophysics	National Institutes of Health*		1	1957
	Biokhimiya		Biochemistry	Consultants Bureau	21	1	1956
Byull. éksp(erim). biol. i med.	Byulleten' éksperimental'noi biologii i meditsiny		Bulletin of Experimental Biology and Medicine	Consultants Bureau	41	1	1959
DAN (SSSR) Dokl(ady) AN SSSR	Doklady Akademii Nauk SSSR		The translation of this journal is published in sections, as follows:				
		Life Sciences	Doklady Biochemistry Section	American Institute of Biological Sciences	106	1	1956
		Life Sciences	Doklady Biological Sciences Sections (Includes: Anatomy, biophysics, cytology, ecology, embryology, endocrinology, evolutionary morphology, genetics, histology, hydrobiology, microbiology, morphology, parasitology, physiology, zoology sections)	American Institute of Biological Sciences	112	1	1957
		Life Sciences	Doklady Botanical Sciences Sections (Includes: Botany, phytopathology, plant anatomy, plant ecology, plant embryology, plant physiology, plant morphology sections)	American Institute of Biological Sciences	112	1	1957
		Chemical Sciences	Proceedings of the Academy of Sciences of the USSR, Section: Chemical Technology	Consultants Bureau	106	1	1956
		Chemical Sciences	Proceedings of the Academy of Sciences of the USSR, Section: Chemistry	Consultants Bureau	106	1	1956
		Chemical Sciences	Proceedings of the Academy of Sciences of the USSR, Section: Physical Chemistry	Consultants Bureau	112	1	1957
		Earth Sciences	Doklady Earth Sciences Sections (Includes: Geochemistry, geology, geophysics, hydrogeology, mineralogy, paleontology, petrography, permafrost sections)	American Geological Institute	124	1	1959
		Earth Sciences	Proceedings of the Academy of Sciences of the USSR, Section: Geochemistry	Consultants Bureau	106-123	1 6	1957-1958
		Earth Sciences	Proceedings of the Academy of Sciences of the USSR, Sections: Geology	Consultants Bureau	106-123	1 6	1957-1958
		Mathematics	Doklady Soviet Mathematics	The American Mathematics Society	131	1	1961
		Physics	Soviet Physics–Doklady (Includes: Aerodynamics, astronomy, crystallography, cybernetics and control theory, electrical engineering, energetics, fluid mechanics, heat engineering, hydraulics, mathematical physics, mechanics, physics, technical physics, theory of elasticity sections)	American Institute of Physics	106	1	1956
		Physics	Proceedings of the Academy of Sciences of the USSR, Applied Physics Sections (does not include mathematical physics or physics sections)	Consultants Bureau	106-117	1	1956-1957
Derevoobrabat. prom-st'.	Derevoobrabatyvayushchaya promyshlennost'		Wood Processing Industry	Timber Development Association (London)		9	1959
	Élecktrosvyaz		Telecommunications	Massachusetts Institute of Technology*		1	1957
Éntom(ol). oboz(renie)	Éntomologicheskoe obozrenie		Entomological Review	American Institute of Biological Sciences	38	1	1959
Farmakol. (i) toksikol(ogiya)	Farmakologiya i toksikologiya		Pharmacology and Toxicology	Consultants Bureau	20	1	1957
FMM	Fizika metallov i metallovedenie		Physics of Metals and Metallography	Acta Metallurgica*	5	1	1957
Fiziol. zhurn. SSSR (im. Sechenova)	Fiziologicheskii zhurnal im. I. M. Sechenova		Sechenov Physiological Journal USSR	National Institutes of Health*		1	1957
Fiziol(ogiya) rast.	Fiziologiya rastenii		Plant Physiology	American Institute of Biological Sciences	4	1	1957
	Geokhimiya		Geochemistry	The Geochemical Society		1	1958
FTT	Fizika tverdogo tela		Soviet Physics—Solid State	American Institute of Physics	1	1	1959
Izmerit. tekh(nika)	Izmeritel'naya tekhnika		Measurement Techniques	Instrument Society of America		1	1959
Izv. AN SSSR, O(td). Kh(im). N(auk)	Izvestiya Akademii Nauk SSSR: Otdelenie khimicheskikh nauk		Bulletin of the Academy of Sciences of the USSR: Division of Chemical Sciences	Consultants Bureau		1	1952

Izmerit. tekh(nika)	Izmeritel'naya tekhnika				1	1952
Izv. AN SSSR, O(td). Kh(im). N(auk)	Izvestiya Akademii Nauk SSSR: Otdelenie khimicheskikh nauk	Bulletin of the Academy of Sciences of the USSR: Division of Chemical Sciences	Consultants Bureau			

continued

Izv. AN SSSR, O(td). T(ekhn). N(auk): Met(all). i top.	(see Met. i top.)					
Izv. AN SSSR Ser. fiz(ich).	Izvestiya Akademii Nauk SSSR: Seriya fizicheskaya	Bulletin of the Academy of Sciences of the USSR: Physical Series	Columbia Technical Translations		1	1954
Izv. AN SSSR Ser. geofiz.	Izvestiya Akademii Nauk SSSR: Seriya geofizicheskaya	Bulletin (Izvestiya) of the Academy of Sciences USSR: Geophysics Series	American Geophysical Union		1	1954
Izv. AN SSSR Ser. geol.	Izvestiya Akademii Nauk SSSR: Seriya geologicheskaya	Izvestiya of the Academy of Sciences of the USSR: Geologic Series	American Geological Institute		1	1958
Kauch. i rez.	Kauchuk i rezina	Soviet Rubber Technology	Research Association of British Rubber Manufacturers	18	1	1959
	Kinetika i kataliz	Kinetics and Catalysis	Consultants Bureau	1	3	1960
	Koks i khimiya	Coke and Chemistry USSR	Coal Tar Research Association (Leeds, England)		1	1958
Kolloidn. zh(urn).	Kolloidnyi zhurnal	Colloid Journal	Consultants Bureau	14	1	1952
	Kristallografiya	Soviet Physics – Crystallography	American Institute of Physics	2	1	1957
Metalov. i term. obrabot. metal.	Metallovedenie i termicheskaya obrabotka metallov	Metal Science and Heat Treatment of Metals	Acta Metallurgica	6	1	1958
	Metallurg	Metallurgist	Acta Metallurgica		1	1957
Met. i top.	Metallurgiya i topliva	Russian Metallurgy and Fuels	Eagle Technical Publications		1	1960
Mikrobiol.	Mikrobiologiya	Microbiology	American Institute of Biological Sciences	26	1	1957
OS	Optika i spektroskopiya	Optics and Spectroscopy	American Institute of Physics		6	1959
	Pochvovedenie	Soviet Soil Science	American Institute of Biological Sciences		1	1958
	Priborostroenie	Instrument Construction	British Scientific Instrument Research Association		1	1959
Pribory i tekhn. éks(perimenta)	Pribory i tekhnika éksperimenta	Instruments and Experimental Techniques	Instrument Society of America		1	1957
Prikl. matem. i mekh.	Prikladnaya matematika i mekhanika	Applied Mathematics and Mechanics	American Society of Mechanical Engineers		1	1958
PTÉ	(see Pribory i tekhn. éks.)					
	Problemy Severa	Problems of the North	National Research Council of Canada	12	1	1957
Radiotekh.	Radiotekhnika	Radio Engineering	Massachusetts Institute of Technology*	2	1	1957
Radiotekh. i élektronika	Radiotekhnika i élektronika	Radio Engineering and Electronics	Massachusetts Institute of Technology*		1	1959
	Stanki i instrument	Machines and Tooling	Production Engineering Research Assoc.		1	1959
	Stal'	Stal (in English)	Iron and Steel Institute	13	1	1956
Stek. i keram.	Steklo i keramika	Glass and Ceramics	Consultants Bureau		4	1959
Svaroch. proiz-vo	Svarochnoe proizvodstvo	Welding Production	British Welding Research Association			
Teor. veroyat. i prim.	Teoriya veroyatnostei i ee primenenie	Theory of Probability and Its Applications	Society for Industrial and Applied Mathematics		1	1956
Tsvet. Metally	Tsvetnye metally	Nonferrous Metals	Primary Sources		1	1960
UFN	Uspekhi fizicheskikh Nauk	Soviet Physics – Uspekhi (partial translation)	American Institute of Physics	66	1	1958
UKh	Uspekhi khimii	Russian Chemical Reviews	The Chemical Society (London)		1	1960
UMN	Uspekhi matematicheskikh nauk	Russian Mathematical Surveys	London Mathematical Society	15	1	1960
Usp. fiz. nauk	(see UFN)					
Usp. khim(ii)	(see UKh)					
Usp. matem. nauk	(see UMN)					
Usp. sovr. biol.	Uspekhi sovremennoi biologii	Russian Review of Biology	Oliver and Boyd		48	1959
Vest. mashinostroeniya	Vestnik mashinostroeniya	Russian Engineering Journal	Production Engineering Research Assoc.		4	1959
Vop. gem. i per. krovi	Voprosy gematologii i perelivaniya krovi	Problems of Hematology and Blood Transfusion	National Institutes of Health*		1	1957
Vop. onk.	Voprosy onkologii	Problems of Oncology	National Institutes of Health*		1	1957
Vop. virusol.	Voprosy virusologii	Problems of Virology	National Institutes of Health*		1	1957
Zav(odsk). lab(oratoriya)	Zavodskaya laboratoriya	Industrial Laboratory	Instrument Society of America	25	1	1959
ZhAKh Zh. anal(it). khimii	Zhurnal analiticheskoi khimii	Journal of Analytical Chemistry USSR	Consultants Bureau	7	1	1952
ZhETF Zh. éksperim. i teor. fiz.	Zhurnal éksperimental'noi i theoreticheskoi fiziki	Soviet Physics–JETP	American Institute of Physics	28	1	1955
ZhFKh Zh. fiz. khimii	Zhurnal fizicheskoi khimii	Russian Journal of Physical Chemistry	The Chemical Society (London)		7	1959
ZhMÉI Zh(urn). mikrobiol. épidemiol. i immunobiol.	Zhurnal mikrobiologii, épidemiologii i immunobiologii	Journal of Microbiology, Epidemiology and Immunobiology	National Institutes of Health*		1	1957
ZhNKh Zh(urn). neorgan(ich). khim(ii)	Zhurnal neorganicheskoi khimii	The Russian Journal of Inorganic Chemistry	The Chemical Society (London)		1	1959
ZhOKh Zh(urn). obshch(ei) khimii	Zhurnal obshchei khimii	Journal of General Chemistry USSR	Consultants Bureau	19	1	1949
ZhPKh Zh(urn). prikl. khimii	Zhurnal prikladnoi khimii	Journal of Applied Chemistry USSR	Consultants Bureau	23	1	1950
ZhSKh Zh(urn). strukt. khimii	Zhurnal strukturnoi khimii	Journal of Structural Chemistry	Consultants Bureau	1	1	1960
ZhTF Zh(urn). tekhn. fiz.	Zhurnal teknicheskoi fiziki	Soviet Physics–Technical Physics	American Institute of Physics	26	1	1956
Zh(urn). vyssh. nervn. deyat. (im. Pavlova)	Zhurnal vysshei nervnoi deyatel'nosti (im. I. P. Pavlova)	Pavlov Journal of Higher Nervous Activity	National Institutes of Health*		1	1958

*Sponsoring organization. Translation through 1960 issues is a publication of Pergamon Press.